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Local Determinacy of Equilibrium Paths in Three-Dimensional New Keynesian Models: An Analysis of the Interaction Between Monetary and Fiscal Policies

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Local Determinacy of Equilibrium Paths in Three-Dimensional New Keynesian Models: An Analysis of the Interaction Between Monetary and Fiscal Policies

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Abstract

To achieve local determinacy of the equilibrium path in a typical New Keynesian model, an active monetary policy must be accompanied by a passive fiscal policy. Furthermore, a passive monetary policy must be accompanied by an active fiscal policy. However, numerous studies have found that this principle does not hold under several additional assumptions. The present study comprehensively identifies a set of monetary and fiscal policy parameters to achieve local determinacy in an orthodox three-dimensional continuous-time New Keynesian model, assuming distortionary taxation. The results demonstrate that, first, when inflation targeting is active in monetary policy implementation, fiscal policy may be ineffective in achieving determinacy if the elasticity of consumption with respect to currency holdings is sufficiently large. Second, implementing output targeting as part of monetary policy can effectively eliminate such cases.

Keywords: New Keynesian (NK) model, inflation targeting, output targeting, government debt targeting, equilibrium determinacy

JEL classification: E52, E61, E62

1 Introduction

This study employs a three-dimensional New Keynesian (NK) model in continuous time that incorporates distortionary taxes (such as income tax) to comprehensively identify the set of parameters for monetary and fiscal policy that leads to the achievement of the local determinacy of equilibrium. The NK model is

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a dynamic general equilibrium model that introduces price stickiness. As an optimization model, it uses equilibrium determinacy as a criterion for assessing system stability. Equilibrium determinacy questions whether the equilibrium paths are unique, that is, whether the optimal selection of an economy's initial state by economic agents is unique.

Specifically, a comparison is made between the number of jump variables (variables for which economic agents are allowed to choose their initial values) in the system and the number of non-convergent roots;[1] if they match, equilibrium determinacy is obtained, at least in the vicinity of the steady-state point. If the number of jump variables exceeds the number of non-convergent roots, the initial states chosen will not be determined uniquely, and there will be multiple equilibrium paths (equilibrium indeterminacy). Conversely, if the number of jump variables is less than the number of non-convergent roots, the system will be unstable. A simple NK model is a dynamic system comprising two endogenous variables: the inflation rate and output (real GDP). Because both are jump variables, two non-convergent roots are required for the local determinacy of the equilibrium path.

The following basic policy positions are derived from this model. Active inflation targeting leads to the local determinacy of equilibrium, while passive inflation targeting leads to indeterminacy (e.g., see Woodford 2003, Chapter 2). "Active" means that the monetary authorities' policy stance is to deal with changes in the rate of inflation by changing nominal interest rates at a rate exceeding 1:1, whereas "passive" means that they respond at a rate of less than 1:1. The policy stance that leads to determinacy is called the Taylor principle.

If the monetary authorities were to incorporate changes in production and inflation rates into their policies, even with a passive stance in relation to changes in the inflation rate, they could achieve local determinacy by responding to these changes in production (Bullard and Mitra 2002). In other words, output targeting could increase the likelihood of local determinacy.[2]

Furthermore, Gerko and Sossounov (2015) demonstrated that when positive trend inflation and capital accumulation are introduced into a typical NK model, an active stance does not necessarily guarantee determinacy (or cause indeterminacy). Similar results were obtained using the model developed by Dupor (2001) who had concluded that determinacy could be achieved through a passive stance toward inflation. Gerko and Sossounov (2015) asserted that the response to the output gap must be positive and sizeable (but not too great) to achieve determinacy under the Taylor principle.

Leeper (1991) developed a framework to analyze the effects of monetary and fiscal policies. This groundbreaking study examined the effects of these inter-

[1]In continuous-time systems, a characteristic root with non-negative real part is a non-convergent root. In discrete-time systems, it is a characteristic root that has an absolute value of one or more.

[2]Carlstrom and Fuerst (2007) developed a model incorporating asset prices as a target variable of the monetary authorities. The effects of asset price targeting differed depending on whether nominal wages or prices are sticky. In the former case, it increases the likelihood of determinacy. In the latter case, it reduces such a likelihood.

actions on equilibrium determinacy. The author postulated a fiscal policy rule (taxation rule) whereby lump-sum taxes are determined by government debt, defining active and passive fiscal policies as follows. Under an active fiscal policy, fiscal authorities manage taxes without being constrained by the government's long-term budgets. Under a passive fiscal policy, taxes are manipulated such that government debt does not diverge over time. Leeper (1991) concluded that achieving local determinacy requires either an active fiscal policy with a passive monetary policy or a passive fiscal policy with an active monetary policy.[3]

Leeper (1991) posited a lump-sum tax as a taxation method. Schmitt-Grohé and Uribe (2007) considered rules for income tax, a distortionary tax, to deal with government debt, confirming the results of Leeper (1991). Kumhof, Nunes, and Yakadina (2010) posited that fiscal authorities might not be able to adequately control tax revenues and expenditures because of government solvency. The authors documented that following the Taylor principle might result in impractical (and even inappropriate) outcomes.[4] However, these studies have not analytically identified any set of monetary and fiscal policy parameters through which local determinacy may be achieved, relying only on numerical calculations instead. Our study presents results that are entirely analytical.

Leith and Thadden (2008) considered the NK model in continuous time with capital accumulation, with the tax collection method being a lump-sum tax. The authors posited that consumers die at a certain rate, which would lead to the economy diverging from a Ricardian equivalence. In their model, the level of government debt strongly influenced local dynamics. In a high-debt (low-debt) situation, an active monetary policy increases (decreases) the degree of fiscal regulation required to ensure determinacy. Thus, the degree of monetary and fiscal controls required to ensure determinacy depends on the target level of government debt.

Alexeeva, Mokaev, and Polshchikova (2020) also considered a non-Ricardian economy (but without capital accumulation). They used a different type of non-linear Phillips function compared from that adopted by Leith and Thadden (2008). Leith and Thadden used a Calvo-type Phillips function, whereas Alexeeva, Mokaev, and Polshchikova used a Rotemberg-type one. As in previous studies, they defined active and passive monetary and fiscal policies to explore the combinations that achieve local determinacy. In their model, determinacy could be achieved in all combinations, except for active monetary and fiscal policies. However, they used only sufficient conditions, not necessary and sufficient conditions, to determine the sign of the characteristic roots. Ac-

[3]When an active monetary policy is combined with a passive fiscal policy, monetary policy must proactively deal with inflation to counter any rise in inflation caused by the passive fiscal policy. Meanwhile, when an active fiscal policy is combined with a passive monetary policy, the inflation allowed by the passive monetary policy works to offset the fiscal situation. This ensures that the economy satisfies transversality conditions, or it will be in a Ricardian regime.

[4]Kumhof, Nunes, and Yakadina (2010) asserted that, assuming interest rates responding to government debt, it is possible to proactively deal with inflation in accordance with the Taylor principle. However, the outcome is unrealistic, as nominal interest rates will breach the 0% minimum.

cordingly, the dynamic nature of the entire plane of coordinates for monetary and fiscal parameters remained unclear unless the parameters were identified. Our study clarifies this using the Routh–Hurwitz and inverse Routh–Hurwitz criteria, which are necessary and sufficient conditions.

Many other studies on equilibrium determinacy have used NK models in continuous time. Benhabib, Schmitt-Grohé, and Uribe (2001) postulated a non-linear interest rate rule for the policies of monetary authorities, under which the nominal interest rate varies exponentially in response to variations in the inflation rate. This resulted in a system with two steady-state points: the target point (spiral) and the low-inflation point (saddle point). Barnett *et al.* (2022) conducted a detailed analysis of the dynamics in the model proposed by Benhabib, Schmitt-Grohé, and Uribe (2001) and demonstrated that, even if the Taylor principle was satisfied, the Shil'nikov chaos attractor could emerge in the vicinity of the targeted steady-state point.[5] This implies that, although the equilibrium is locally determinate, it could be indeterminate from a global perspective. Furthermore, Barnett *et al.* (2023) considered policy options for controlling the Shil'nikov chaos.[6] However, differently from these works, our study aims to clarify the local uniqueness (determinacy) of the equilibrium path rather than attempting to consider it in a global context.

This study analyzes the local equilibrium determinacy using a continuous-time three-dimensional NK model that assumes income taxation to be the taxation method. The model features three endogenous variables: inflation rate, output, and government debt. Inflation rate and output are jump variables, and government debt is a predetermined variable (a variable with a given initial state). Thus, two non-convergent roots are required to achieve local determinacy. We posit a monetary policy rule to adjust nominal interest rates in response to fluctuations in the inflation rate and output and a fiscal policy rule for adjusting income tax rates in response to government debt. We primarily consider two cases. The first, the base case, involves monetary authorities targeting only the inflation rate (cases in which the coefficient for dealing with output is zero). In this case, a situation can emerge in which an active monetary policy cannot achieve local determinacy regardless of the implemented fiscal policy, that is, the fiscal policy becomes ineffective. The second scenario considers a situation in which the monetary authorities consider policy in view of both the rate of inflation and the volume of output. In this case, if output

[5]Benhabib, Schmitt-Grohé and Uribe (2002) demonstrated the chaos occurrence, but failed to identify it as Shil'nikov chaos. Barnett *et al.* (2022) used the Shil'nikov criterion (a criterion for confirming the existence of a fractal attractor set) to identify chaos.

[6]Specifically, Barnett *et al.* (2023) considered open-loop control methods for controlling chaos while preserving the feedback effects inherent in monetary policy rules (interest rate rules). Open-loop control means the following. First, announce (commit to) a high nominal interest rate achieved at a targeted steady-state point. This announcement results in inflationary expectations, driving the economy to the targeted point, thereby establishing a long-term "anchor" for inflation. Consequently, the targeted point can be achieved within a relatively short period of time (although chaos remains). This method is the Ott, Grebogi, and Yorke (OGY) algorithm and a well-established technique for controlling chaos, particularly in the field of engineering.

targeting is sufficiently aggressive, then fiscal policies will remain effective. This new finding sheds light on a different facet of output targeting that previous research has not explored, demonstrating its effectiveness.

In our model, a passive fiscal policy within an active monetary policy regime is the only necessary condition for local determinacy. Additionally, an active fiscal policy within a passive monetary policy regime constitutes a sufficient condition for local determinacy. This is another major difference from the conventional NK models.

The remainder of this paper is organized as follows. Section 2 presents the NK model, which serves as the foundation of our analysis. Section 3 examines the case of "pure" inflation targeting. Section 4 presents a similar analysis that incorporates output into monetary authorities' target variables, and Section 5 concludes the paper.

2 Model

Following Benhabib, Schmitt-Grohé, and Uribe (2003), we present a continuous-time three-dimensional NK model. The modeled economy comprises three types of entities: household–firm units continuously distributed in the interval [0, 1] (the private sector), monetary authorities, and fiscal authorities. Household–firm units produce differentiated goods that are put together (assembled) and consumed as final goods. The monetary authorities set nominal interest rates, and the fiscal authorities set income tax rates. This model can be summed up in terms of three differential equations: the consumption Euler equation, the NK Phillips function, and the public sector's intertemporal budget constraint equation. We begin by describing the behavior of each economic entity.

2.1 Private sector

Household–firm units assigned by the number of $j \in [0,\, 1]$ aim to maximize the value of the cumulative utility U_j achievable from the present moment extending into eternity ($t \in [0,\, \infty)$), expressed as

$$U_j := \int_0^\infty e^{-\rho t} u_j(t) dt, \tag{1}$$

where $u_j(t)$ is the instantaneous utility function, and $\rho > 0$ is the subjective discount rate. Function $u_j(t)$ is expressed as

$$u_j(t) = \log\left(c_j(t)^\sigma m_j(t)\right) - \frac{\ell_j(t)^{1+\psi}}{1+\psi} - \frac{\eta}{2}\left(v_j(t) - v_j^*\right)^2, \tag{2}$$

where $c_j(t)$ is the consumption of final goods, $m_j(t)$ is the real money balances, $\ell_j(t)$ is the labor supply, $v_j(t)$ is the rate of change in the price of good j, $\sigma > 0$ is the reciprocal of the elasticity of consumption with respect to currency holdings, $\psi > 0$ is the elasticity of labor's marginal disutility, and $\eta > 0$ is the scale of price revision costs. Price revision costs are a psychological burden on producers because of price negotiations and similar factors.[7] To simplify our calculations, we express price revision costs not as an actual rate of change $v_j(t)$, but rather in relation to the size of its deviation from the steady-state value v_j^*. However, this assumption does not affect our conclusions. Revision costs create stickiness in prices; therefore, we can view η as a measure of the degree of price stickiness.

Final goods are manufactured by assembling differentiated goods. Accordingly, differentiated goods can be viewed as intermediate goods in the production of final goods. We assume that manufacturing final goods requires only the intermediate goods of which they are composed, and no other factors of production are needed,[8] expressed by the constant elasticity of substitution (CES) function:

$$y(t) = \left[\int_0^1 y_j(t)^\alpha dj \right]^{\frac{1}{\alpha}}, \tag{3}$$

where $y(t)$ denotes the production amount of final goods, $y_j(t)$ denotes the input of intermediate good j, and $0 < \alpha < 1$. The elasticity of substitution among intermediate goods is expressed as $\phi := 1/(1-\alpha) > 1$. At each point in time, the producers of final goods with technology (3), taking the price of good j, which we denote $p_j(t)$, as given, determine the volume of demand for these goods to minimize costs $\int_0^1 p_j(t) y_j(t) dj$.[9] Under optimal conditions, the demand function for good j is expressed as

$$y_j(t) = \left(\frac{p_j(t)}{p(t)} \right)^{-\phi} y(t), \tag{4}$$

where $p(t)$ represents a price indicator defined as

$$p(t) = \left[\int_0^1 p_j(t)^{1-\phi} dj \right]^{\frac{1}{1-\phi}}.$$

Next, we consider the budget constraints for household–firm units. The total assets $A_j(t)$ of household–firm unit j are composed of the stock of nominal money $M_j(t)$ and nominal bonds (government bonds) $B_j(t)$, or $A_j(t) = M_j(t) + B_j(t)$. In this case, the intertemporal budget constraint equation is expressed as $\dot{A}_j(t) = (1-\tau(t)) p_j(t) y_j(t) + R(t) B_j(t) - p(t) c_j(t)$, where $R(t)$ denotes the nominal interest rate for bonds and $\tau(t) < 1$ denotes the income tax rate. This equation can be rewritten in real terms as

$$\dot{a}_j(t) = (1-\tau(t)) \frac{p_j(t)}{p(t)} y_j(t) + r(t) a_j(t) - c_j(t) - R(t) m_j(t), \tag{5}$$

[7] This equation treating price revision costs as a psychological burden will greatly simplify subsequent calculations. Similar equations have also been used by Benhabib, Schmitt-Grohé and Uribe (2003), and Carlstrom and Fuerst (2007). The specification to a quadratic function follows Rotemberg (1982).

[8] This type of assumption is often seen in R&D-based endogenous growth models (e.g., Grossman and Helpman 1991).

[9] This is the so-called isoperimetric problem. See Chapter 6 in Chiang (1992) for the solution.

where $a_j(t) := A_j(t)/p(t)$ is the real value of assets, and $r(t) := R(t) - \dot{p}(t)/p(t)$ is the real interest rate.

Assume that producing one unit of any good j requires one unit of labor,

$$y_j(t) = \ell_j(t). \tag{6}$$

Under this production technology, the household–firm units facing the demand function (4) choose paths $c_j(t)$, $m_j(t)$, and $v_j(t)$ to maximize (1). The constraints are given in (5) and the transition equation for price $p_j(t)$ is expressed as

$$\dot{p}_j(t) = v_j(t)p_j(t). \tag{7}$$

In addition, the initial conditions $a_j(0) > 0$ and $p_j(0) > 0$ are considered given. From the optimality conditions, we derive the following (see Appendix A.1).

$$\dot{c}_j(t) = [R(t) - v(t) - \rho]\, c_j(t), \tag{8}$$

$$\dot{v}_j(t) = \rho\left(v_j(t) - v_j^*\right) - \frac{\phi}{\eta} y_j(t)^{1+\psi} + \frac{\sigma(\phi - 1)}{\eta}(1 - \tau(t))\frac{p_j(t)y_j(t)}{p(t)c_j(t)}, \tag{9}$$

$$m_j(t) = \frac{c_j(t)}{\sigma R(t)}. \tag{10}$$

Equation (8) is the consumption Euler equation, (9) is the NK Phillips function, and (10) is the money demand function.

Because all household–firm units behave in accordance with the above equations (symmetry among household–firm units), we can drop subscript j from all variables. In addition, because $j \in [0, 1]$, we can regard them as aggregate variables.

2.2 Public sector

The intertemporal budget constraint equation for the public sector is expressed as $\dot{B}(t) = R(t)B(t) - \dot{M}(t) - \tau(t)p(t)y(t) + p(t)g(t)$, where $g(t)$ denotes real government expenditure. This equation describes a fiscal regime (the so-called Ricardian regime) in which the funds for paying back an increased volume of government bonds will be covered by higher future taxes and seigniorage. This equation can be rewritten in real terms as[10]

$$\dot{a}(t) = r(t)a(t) - R(t)m(t) - \tau(t)y(t) + g(t). \tag{11}$$

To simplify the discussion, we assume that real government expenditure $g(t)$ is a certain percentage of production $y(t)$:[11]

$$g(t) = \beta y(t), \tag{12}$$

where $0 < \beta < 1$.

Finally, we present the goods market clearing condition (equilibrium condition):

$$y(t) = c(t) + g(t). \tag{13}$$

[10]In symmetric equilibrium, Equation (11) for public-sector budget constraint is equivalent to Equation (5) for household–firm units' budget constraint.

[11]This assumption is based on the study by Shinagawa and Tsuzuki (2019).

2.3 Dynamic system

Equations (8) to (13) can be summarized in the following three differential equations.

$$\begin{aligned}\dot{y}(t) &= [R(t) - v(t) - \rho]\, y(t),\\ \dot{v}(t) &= \rho\,[v(t) - v^*] - \frac{\phi}{\eta} y(t)^{1+\psi} + \frac{\sigma\,(\phi - 1)}{\eta\,(1-\beta)}\,[1 - \tau(t)]\,,\\ \dot{a}(t) &= [R(t) - v(t)]\, a(t) - \frac{1-\beta}{\sigma} y(t) - \tau(t) y(t) + \beta y(t).\end{aligned} \tag{14}$$

2.4 Policy rules

To close the dynamic System (14), we formulate the monetary and fiscal authorities' policy rules. We assume that the objective of each authority is to stabilize the economy. To this end, they change nominal interest and tax rates in response to the target variables' deviations from their steady-state values.[12]

2.4.1 Interest rate rule

We assume that the target variable for the monetary authorities is the inflation rate (inflation targeting policy). Thus, the monetary policy rule can be expressed as

$$R(t) = \bar{R} + R_v\,[v(t) - v^*]\,, \tag{15}$$

where v^* is the target value of the inflation rate (= its steady-state value), $\bar{R} > 0$ is the nominal interest rate by which it is achieved, and $R_v > 0$ is the policy parameter that expresses the extent to which the nominal interest rate reacts to changes in the rate of inflation. Using terminology from Leeper (1991), we term $R_v > 1$ as an active monetary policy and $R_v < 1$ as a passive one.

2.4.2 Tax rate rule

We posit the fiscal policy (taxation) rule for dealing with the real government debt $a(t)$ by manipulating the income tax rate $\tau(t)$:

$$\tau(t) = \bar{\tau} + \tau_a\,[a(t) - a^*]\,, \tag{16}$$

where a^* is the target real government debt (= steady-state value of $a(t)$), $\bar{\tau} \in (0, 1)$ is the income tax rate by which it is achieved, and $\tau_a > 0$ is the policy parameter governing the extent to which the income tax rate reacts to changes in the government debt. Following Schmitt-Grohé and Uribe (2007), we term $\tau_a < r/y$ an active fiscal policy and $\tau_a > r/y$ a passive one.

Substituting (16) in (11) we obtain $\dot{a}(t) = r(t)a(t) - R(t)m(t) - \{\bar{\tau} + \tau_a\,[a(t) - a^*]\}y(t) + g(t)$. Therefore, an active fiscal policy means $\partial\dot{a}(t)/\partial a(t) >$

[12] In this analysis, besides excluding the appearance of multiple steady-state points, our major goal is to clarify the local dynamics around a steady-state point, and we use policy rules specified in linear functions.

0, and a passive one means $\partial \dot{a}(t)/\partial a(t) < 0$. Thus, whether $\tau_a < r/y$ or $\tau_a > r/y$ represents the differences in government's adoption of relaxed or disciplined fiscal management.

3 Local determinacy of equilibrium

This section analyzes the local determinacy of the dynamic path around the steady-state point of the macroeconomic system derived from Equations (14) to (16).

3.1 Characteristic equation

If we substitute policy rules (15) and (16) into (14), we obtain the following differential equation system that contains $y(t)$, $v(t)$, and $a(t)$ as endogenous variables.

$$\begin{aligned} \dot{y}(t) &= \left[\bar{R} + R_v \{v(t) - v^*\} - v(t) - \rho\right] y(t), \\ \dot{v}(t) &= \rho \left[v(t) - v^*\right] - \frac{\phi}{\eta} y(t)^{1+\psi} + \frac{\sigma(\phi - 1)}{\eta(1-\beta)} \left[1 - \bar{\tau} - \tau_a \{a(t) - a^*\}\right], \\ \dot{a}(t) &= \left[\bar{R} + R_v \{v(t) - v^*\} - v(t)\right] a(t) - \frac{1-\beta}{\sigma} y(t) \\ &\quad - \left[\bar{\tau} + \tau_a \{a(t) - a^*\}\right] y(t) + \beta y(t). \end{aligned} \tag{17}$$

The non-trivial steady-state values of this system can be obtained as follows.

$$y^* = \left[\frac{\sigma(1-\bar{\tau})(\phi - 1)}{\phi(1-\beta)}\right]^{\frac{1}{1+\psi}}, \quad v^* = \bar{R} - \rho, \quad a^* = \frac{1}{\rho}\left(\frac{1-\beta}{\sigma} + \bar{\tau} - \beta\right) y^*. \tag{18}$$

To ensure that $a^* > 0$, we make the following assumption.

Assumption 3.1 $\frac{1-\beta}{\sigma} + \bar{\tau} - \beta > 0$.

This implies that the primary balance is in surplus in the steady state. Using (10), (12) and (13), the expression $\left(\frac{1-\beta}{\sigma} + \bar{\tau} - \beta\right) y^*$ can be rewritten as

$$\left(\frac{1-\beta}{\sigma} + \bar{\tau} - \beta\right) y^* = \bar{R}m^* + \bar{\tau}y^* - g^* = \frac{1}{p}\left(\bar{R}M + \bar{\tau}py^* - pg^*\right).$$

For the central bank to supply the money stock by M, the same amount of government bonds must have been issued by M. Thus, M equals the central bank's government bond holdings. $\bar{R}M$ is the central bank's interest income, which ultimately becomes government revenue as payment to the treasury. Thus, $\bar{R}M + \bar{\tau}py^* - pg^*$ represents the primary balance. Assumption 3.1 ensures this is in surplus. If the primary balance is in deficit, then interest payments on government bonds ($\bar{R}A$) are added to the expenditure, and fiscal balance cannot be achieved. After considering the reduction in real government debt outstanding due to inflation, a surplus primary balance and an equal amount of government debt expenditure result in a zero increase or decrease in $a(t)$, that is, a steady state.[13]

The Jacobian matrix of System (17) evaluated at the steady-state point is expressed as

$$J = \begin{bmatrix} 0 & (R_v - 1)\, y^* & 0 \\ -P_1 & \rho & -P_2 \tau_a \\ -\rho a^*/y^* & (R_v - 1)\, a^* & \rho - \tau_a y^* \end{bmatrix},$$

where $P_1 = \frac{\phi(1+\psi)}{\eta}(y^*)^{\psi} > 0$ and $P_2 = \frac{(\phi-1)\sigma}{(1-\beta)\eta} > 0$. We restrict our analysis to cases where the matrix J is nonsingular (i.e., det $J \neq 0$), thereby excluding the case of $R_v = 1$.[14]

The characteristic equation of System (17) is expressed as

$$\Delta(\lambda) := |\lambda I - J| = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0, \tag{19}$$

where λ is a characteristic root and I is the unit matrix,

$$b_1 = -\mathrm{tr}\, J = \rho\,(\delta - 2)\,, \tag{20}$$

$$b_2 = |J_{11}| + |J_{22}| + |J_{33}| = \rho^2\,[1 - \delta + \kappa\,(\omega + \delta)]\,, \tag{21}$$

$$b_3 = -\det\, J = \rho^3\kappa\,[\delta\,(\omega - 1) - \omega]\,. \tag{22}$$

Here we used the following notations.

$$\delta := \frac{y^*}{\rho}\tau_a,\ \kappa := (R_v - 1)\,\frac{P_2 a^*}{\rho y^*},\ \omega := \frac{P_1 {y^*}^2}{P_2 \rho a^*}.$$

In addition, $|J_{ij}|$ signifies the minor determinant obtained by removing row i and column j from $|J|$ and the following holds.

$$b_1 b_2 - b_3 = \rho^3\left[\left(\delta^2 - \delta - \omega\right)\kappa - \left(\delta^2 - 3\delta + 2\right)\right]. \tag{23}$$

[13] Assumption 3.1 is a condition that guarantees $a^* > 0$, but realistically, it would need to guarantee $B/p > 0$ in the steady state. ($B/p < 0$ implies loans from the government to the private sector.) Using (10), (12), (13) and (17), we obtain the following equation.

$$a^* - m^* = \left(\frac{B}{p}\right)^* = \frac{1}{\rho}\left[\frac{1-\beta}{\sigma}\left(1 - \frac{\rho}{\bar{R}}\right) + \bar{\tau} - \beta\right] y^*.$$

If this is positive, then Assumption 3.1 is naturally satisfied. If β is set to a reasonable value of 0.2, and ρ, $\bar{R}$, and $\bar{\tau}$ are set to 0.005, 0.015, and 0.12, respectively, following Tsuzuki (2016), then $(B/p)^* > 0$ holds for $\sigma < 6.6$. Therefore, if the elasticity of consumption with respect to money holdings ($1/\sigma$) is 0.15 or higher, Assumption 3.1 is satisfied (as a sufficient condition). In general, the elasticity of consumption with respect to money holdings assumes a value close to 1, meaning that this assumption is almost certainly satisfied.

[14] The purpose of the assumption det $J \neq 0$ is to eliminate the occurrence of characteristic root 0. Rather than being a necessity from an economic perspective, this assumption is made for the analytical requirements of our discussion. Without this assumption, it would be difficult to determine the sign of the characteristic roots.

The active/passive monetary policy corresponds to the positive/negative of κ, that is,

$$\kappa > 0 \Longleftrightarrow R_v > 1 \text{ (active)},$$
$$\kappa < 0 \Longleftrightarrow R_v < 1 \text{ (passive)}.$$

The active/passive fiscal policy in the steady state corresponds to whether δ is less than or greater than 1:

$$\delta < 1 \Longleftrightarrow \tau_a < \rho/y^* \text{ (active)},$$
$$\delta > 1 \Longleftrightarrow \tau_a > \rho/y^* \text{ (passive)}.$$

Our objective is to determine the signs of the roots of characteristic Equation (19) under different combinations of these policies.

3.2 Conditions related to the signs of characteristic roots

In System (17), $c(t)$ and $v(t)$ are jump variables and $a(t)$ is a predetermined variable (state variable).[15] Therefore, if there are two characteristic roots with positive real parts, the equilibrium is locally determinate. Let λ_1, λ_2, λ_3 represent the three characteristic roots. From the correlation between the roots and the coefficients ($b_3 = -\lambda_1\lambda_2\lambda_3$), if $b_3 > 0$, the signs of the three roots are $(+, +, -)$ or $(-, -, -)$, and conversely, if $b_3 < 0$, they are $(+, +, +)$ or $(+, -, -)$.[16]

Furthermore, in cases in which $b_3 > 0$, we can use the Routh–Hurwitz criterion. This criterion provides the necessary and sufficient conditions for the real parts of all roots to be negative.

(RH-I) $b_1 > 0$,

(RH-II) $b_3 > 0$,

(RH-III) $b_1 b_2 - b_3 > 0$.

When $b_3 > 0$, if either or both (RH-I) and (RH-III) are *not* satisfied, the signs of the roots are $(+, +, -)$, so that the equilibrium is locally determinate. If both are satisfied, the signs are $(-, -, -)$, resulting in indeterminacy (degree 2).

Simultaneously, when $b_3 < 0$, we can use the necessary and sufficient conditions for the real parts of all roots to be positive. We term these the inverse Routh–Hurwitz criterion (see Appendix A.2 for proof of this criterion).

(IRH-I) $b_1 < 0$,

(IRH-II) $b_3 < 0$,

(IRH-III) $b_1 b_2 - b_3 < 0$.

When $b_3 < 0$, if either or both (IRH-I) and (IRH-III) are not satisfied, the

[15] Basically, the variables whose initial conditions (boundary conditions) come from outside the system are state variables. Variables that are allowed to (optimally) choose their initial conditions to satisfy the transversality conditions are jump variables.

[16] The case in which $b_3 = 0$ (when one of the roots is zero) has been excluded by the assumption of $\det J \neq 0$.

signs of the three roots are $(+, -, -)$, and the equilibrium path is locally indeterminate (degree 1). If they are satisfied, the signs are $(+, +, +)$ and the equilibrium is unstable.

For an equilibrium to be determinate, the condition $b_3 > 0$ must first be satisfied. However, from (22), we know that if $\omega > 1$ does not hold true, in the case of $\kappa > 0$, a positive δ that yields $b_3 > 0$ does not exist. In other words, if the inequality $\omega > 1$ does not hold, it is impossible to achieve determinacy under an active monetary policy, regardless of the fiscal policy adopted; yet, this does not appear realistic. To ensure the validity of the model, then, we make the following assumption.[17]

Assumption 3.2 $\omega > 1$.

We explore the type of fiscal policy needed to achieve local determinacy when the monetary policy is active or passive under Assumptions 3.1 and 3.2.

3.3 Active monetary policy

First, we consider an active monetary policy. Assuming $\kappa > 0$, we determine whether the Routh–Hurwitz criterion or its inverse is satisfied. To this end, we examine the signs of b_3, b_1, and $b_1 b_2 - b_3$.

3.3.1 The sign of b_3

With Assumption 3.2, we derive the following relations from (22).[18]

$$b_3 \lesseqgtr 0 \iff \delta \lesseqgtr \frac{\omega}{\omega - 1}. \tag{24}$$

From (24), if $\delta < \omega/(\omega - 1)$, we get $b_3 < 0$. Accordingly, the signs of the three characteristic roots are either $(+, +, +)$ or $(+, -, -)$. In other words, the equilibrium is unstable or indeterminate (degree 1). Meanwhile, if $\delta > \omega/(\omega - 1)$, then $b_3 > 0$, so the signs of the roots are either $(+, +, -)$ or $(-, -, -)$. Therefore, in this case, the equilibrium is either determinate or indeterminate (degree 2).

Next, we consider the sign of b_1.

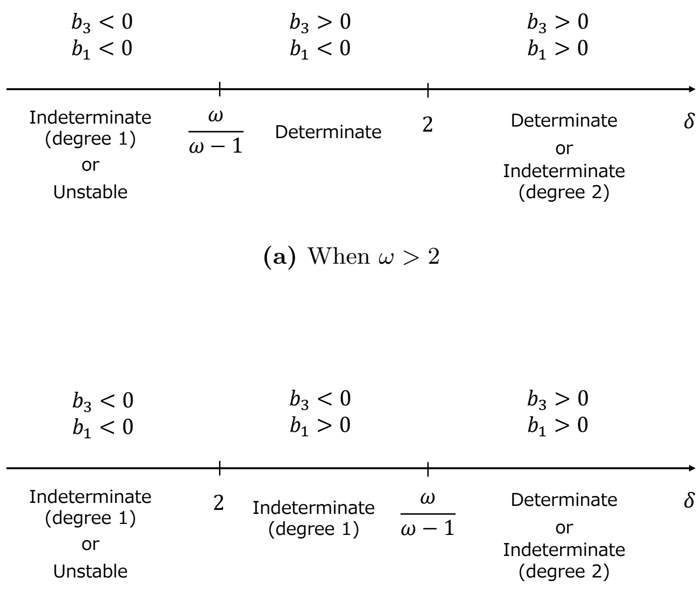
3.3.2 The sign of b_1

From (20), we derive the following correlations.

$$b_1 \lesseqqgtr 0 \iff \delta \lesseqqgtr 2. \tag{25}$$

[17]If $\sigma > 1$ and $\bar{\tau} < \beta$, then $\omega > 1$ holds. $\sigma > 1$ implies that the utility of consumption exceeds that of money holdings. $\bar{\tau} < \beta$ implies that taxation does not exceed government spending in the steady state. These conditions are sufficient to satisfy Assumption 3.2. With the numerical example presented in Footnote 13, Assumption 3.2 is fulfilled for $\sigma > 10/23 \approx 0.435$.

[18]The case in which $\delta = \omega/(\omega - 1)$ is excluded from consideration by the assumption of $\det J \neq 0$.



(a) When $\omega > 2$

(b) When $\omega < 2$

Figure 1 The signs of b_3 and b_1 under active monetary policy

From (24) and (25), we know that the signs of the characteristic roots depend on whether ω is larger or smaller than 2. If $\omega > 2$, then $\omega/(\omega - 1) < 2$. Thus, the combination of the signs of b_3 and b_1 can have three patterns, as shown in Figure 1(a). At the same time, if $\omega < 2$, then $\omega/(\omega - 1) > 2$, which means the combinations of the signs of b_3 and b_1 are those in Figure 1(b). As for $\omega = 2$, this can be analyzed as a special case of $\omega < 2$ (explained in Appendix A.3).

When $b_3 > 0$ and $b_1 \leq 0$ (including the rightmost point) in Figure 1(a), the Routh–Hurwitz criterion is not satisfied, the signs of the characteristic roots are $(+, +, -)$, and the equilibrium is locally determinate. In addition, when $b_3 < 0$ and $b_1 \geq 0$ (including the leftmost point) in Figure 1(b), the inverse Routh–Hurwitz criterion is not satisfied, the signs of the characteristic roots are $(+, -, -)$, and the equilibrium is indeterminate (degree 1).

In both Figures 1(a) and 1(b), when $b_3 < 0$ and $b_1 < 0$, the signs of the three characteristic roots remain undetermined, that is, they are either $(+, -, -)$ or $(+, +, +)$. Therefore, the equilibrium is either indeterminate (degree 1) or unstable. In addition, when $b_3 > 0$ and $b_1 > 0$, the signs of the characteristic roots remain undetermined. They are either $(+, +, -)$ or $(-, -, -)$, and hence, equilibrium is determinate or indeterminate (degree 2).

We now consider the sign $b_1 b_2 - b_3$ to clarify the signs of the characteristic roots in these two cases.

3.3.3 The sign of $b_1 b_2 - b_3$

We first consider $b_1 b_2 - b_3 = 0$, which is the boundary between positive and negative for $b_1 b_2 - b_3$.

By solving the equation $b_1 b_2 - b_3 = 0$ for κ using (23), we obtain the following.

$$\kappa = \kappa^*(\delta) := \frac{\delta^2 - 3\delta + 2}{\delta^2 - \delta - \omega}. \tag{26}$$

Drawing this function on δ-κ plane gives us the result depicted in Figure 2.[19] The shapes of the curves differ greatly depending on whether $\omega > 2$ or $\omega < 2$. Figure 2(a) presents the case in which $\omega > 2$, and Figure 2(b) presents the case in which $\omega < 2$. First, we will explain how these figures are rendered. Then, we add in the results from the previous paragraphs to identify a locally determinate region on the δ-κ plane in each case of $\omega > 2$ and $\omega < 2$.

The sign of $\kappa^{*\prime}(\delta)$ is the exact same sign as the following quadratic function if the denominator of (26) is non-zero.

$$f(\delta) := \delta^2 - (\omega + 2)\,\delta + \left(1 + \frac{3}{2}\omega\right).$$

The solutions to $f(\delta) = 0$ are as follows.

$$\delta^*_{\pm} = \frac{1}{2}\left(\omega + 2 \pm \omega\sqrt{1 - \frac{2}{\omega}}\right). \tag{27}$$

Accordingly, we can derive the following.

(a) When $\omega > 2$, $\kappa^*(\delta)$ is an increasing function for $[0, \delta^*_-)$ and (δ^*_+, ∞). For (δ^*_-, δ^*_+), it is a decreasing function.

(b) When $\omega < 2$, $f(\delta) = 0$ has no real solution, and $f(\delta) > 0$ for all $\delta > 0$. Therefore, for all $\delta > 0$, $\kappa^*(\delta)$ is an increasing function.

Furthermore, the asymptote parallel to the vertical axis (κ axis) of the fractional function (26) can be derived as follows.

$$\delta = \hat{\delta} := \frac{1}{2}\left(1 + \sqrt{1 + 4\omega}\right). \tag{28}$$

When $\omega > 2$, $\delta^*_- < \hat{\delta} < \delta^*_+$ holds true because $f(\hat{\delta}) < 0$.[20] Since there are at the most two δ satisfying $\kappa^*(\delta) = C$ (C is an arbitrary constant), definitely

[19] We have drawn the graph for both $\kappa > 0$ and $\kappa < 0$ for later discussion.

[20] We show that if $\omega > 2$, then $f(\hat{\delta}) < 0$, and if $\omega < 2$, then $f(\hat{\delta}) > 0$. From $\hat{\delta}^2 = \hat{\delta} + \omega$, we derive the following.

$$\begin{aligned} f(\hat{\delta}) &= \hat{\delta} + \omega - (\omega + 2)\,\hat{\delta} + \left(1 + \frac{3}{2}\omega\right) \\ &= -(\omega + 1)\,\hat{\delta} + \left(1 + \frac{5}{2}\omega\right). \end{aligned}$$

$\kappa^*(\delta_+^*) \geq \kappa^*(\delta_-^*)$.[21] In addition, $\kappa^*(0) = -2/\omega$, $\lim_{\delta\to\infty} \kappa^*(\delta) = 1$, and $\kappa^*(1) = \kappa^*(2) = 0$, and when $\omega > 2$, $1 < \delta_-^* < 2$, and when $\omega < 2$, $1 < \hat{\delta} < 2$.

From the above discussion, in cases of $\omega > 2$ and $\omega < 2$, function (26) is depicted as in Figures 2(a) and 2(b), respectively.

Next, we find the sign of $b_1b_2 - b_3$ in each area in Figures 2(a) and 2(b). For $\delta < \hat{\delta}$ (on the left side of the asymptote), $\delta^2 - \delta - \omega < 0$ is valid from (26). In this case, from (23), $b_1b_2 - b_3$ is decreasing with respect to κ. Therefore, $b_1b_2 - b_3 < 0$ for $\kappa > \kappa^*(\delta)$ and $b_1b_2 - b_3 > 0$ for $\kappa < \kappa^*(\delta)$. As for $\delta > \hat{\delta}$ (on the right side of the asymptote), the reverse holds true. Thus, we have:

- If $\delta < \hat{\delta}$ and $\kappa < \kappa^*(\delta)$, or $\delta > \hat{\delta}$ and $\kappa > \kappa^*(\delta)$, then $b_1b_2 - b_3 > 0$.
- If $\delta < \hat{\delta}$ and $\kappa > \kappa^*(\delta)$, or $\delta > \hat{\delta}$ and $\kappa < \kappa^*(\delta)$, then $b_1b_2 - b_3 < 0$.

From this discussion, as in Figure 2, we determine the sign of $b_1b_2 - b_3$.

Combining Figures 1 and 2, the threshold of δ in Figure 1, which is $\omega/(\omega-1)$, satisfies $1 < \omega/(\omega-1) < 2$ when $\omega > 2$, and $\omega/(\omega-1) > 2$ when $\omega < 2$. The relationship between δ_-^* and $\omega/(\omega-1)$ is not generally settled.[22] However, this does not affect our discussion. With these considerations, by combining the results from Figures 1 and 2, we can specify the local determinacy of equilibrium in the area of $\kappa > 0$ on the δ-κ plane.

In the case of $b_3 < 0$ and $b_1 < 0$ ($\delta < \omega/(\omega-1)$) in Figure 1(a), if $b_1b_2 - b_3 < 0$, the inverse Routh–Hurwitz criterion is satisfied, and the signs of the roots are $(+, +, +)$, and the equilibrium is unstable. However, if $b_1b_2 - b_3 \geq 0$, the signs of the roots are $(+, -, -)$, which implies that the equilibrium is indeterminate (degree 1). At the same time, when $b_3 > 0$ and $b_1 > 0$ ($\delta > 2$) in Figure 1(a), if $b_1b_2 - b_3 > 0$, the Routh–Hurwitz criterion is satisfied and the signs of the roots are $(-, -, -)$, which implies indeterminacy (degree 2). Conversely, if $b_1b_2 - b_3 \leq 0$, then $(+, +, -)$, which implies that the equilibrium is locally determinate. The first quadrant of Figure 3(a) summarizes the results discussed above. Under the assumption that monetary policy is active ($\kappa > 0$), when $\omega > 2$, the region of local determinacy is S_{a}^a in Figure 3(a), which is represented by the shaded area.

Likewise, in the case of $\omega < 2$, combining the results from Figure 1(b) with Figure 2(b) and applying Routh–Hurwitz criterion and its inverse, we can

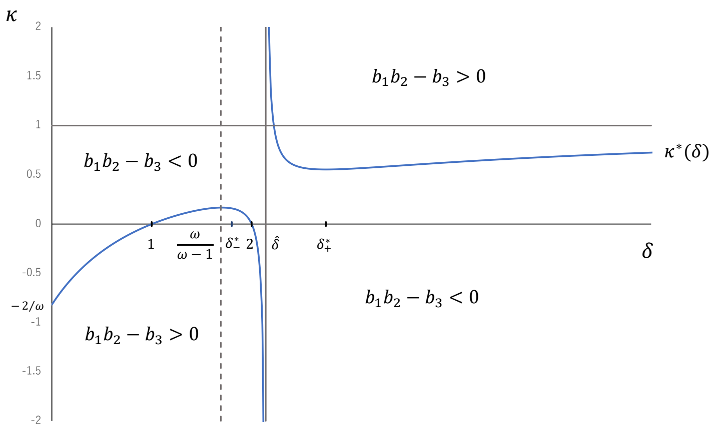
Substituting (28) in the above, we get:

$$\begin{aligned} f(\hat{\delta}) &= -(\omega+1)\cdot\frac{1}{2}\left(1+\sqrt{1+4\omega}\right) + \left(1+\frac{5}{2}\omega\right) \\ &= \frac{1}{2}\left[1+4\omega-(1+\omega)\sqrt{1+4\omega}\right] \\ &= \frac{1}{2}\sqrt{1+4\omega}\left[\sqrt{1+4\omega}-(1+\omega)\right]. \end{aligned}$$

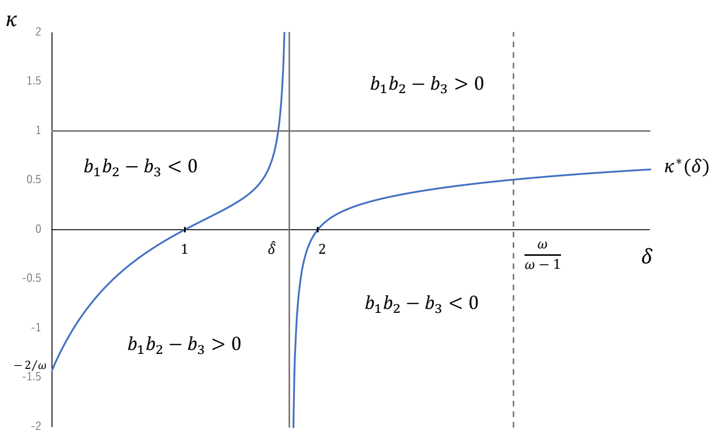
Therefore, $f(\hat{\delta}) \lesseqgtr 0 \iff 1+4\omega \lesseqgtr (1+\omega)^2$, that is, $f(\hat{\delta}) \lesseqgtr 0 \iff \omega \gtreqless 2$.

[21] The values of δ satisfying $\kappa^*(\delta) = C$ are solutions to a certain quadratic equation. If $\kappa^*(\delta_+^*) < \kappa^*(\delta_-^*)$, then there would exist four δ satisfying $\kappa^*(\delta) = C$ for $C \in \left(\kappa(\delta_+^*), \kappa(\delta_-^*)\right)$. This is a contradiction.

[22] If $\omega \lesseqgtr 2.4142$, then $\delta_-^* \lesseqgtr \omega/(\omega-1)$.

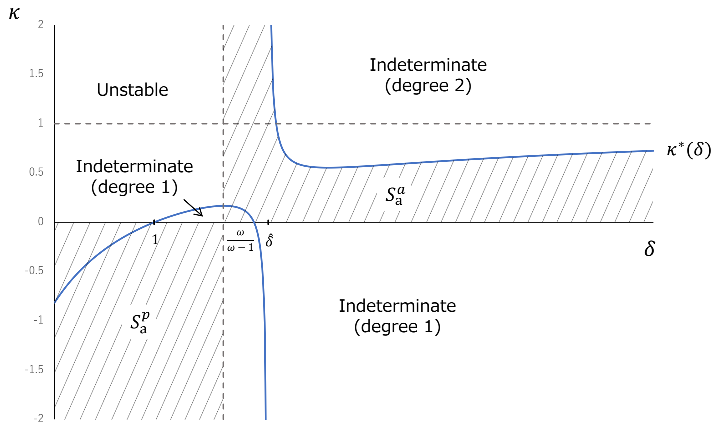


(a) When $\omega > 2$

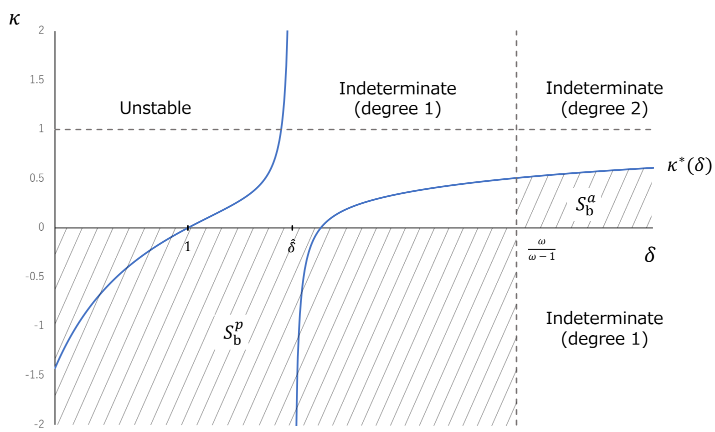


(b) When $\omega < 2$

Figure 2 Curves depicting $b_1 b_2 - b_3 = 0$



(a) When $\omega > 2$



(b) When $\omega < 2$

Figure 3 Locally determinate areas under conditions of pure inflation targeting determine the local determinacy of equilibrium in each area of the δ-κ plane, as in Figure 3(b) (look only at the first quadrant). In this figure, the region of local determinacy is indicated by the shaded portion S_{b}^{a}.

We summarize the results of the above analysis in the form of a proposition.

Proposition 3.1 *Under Assumptions* 3.1 *and* 3.2 *and an active monetary policy, the equilibrium of System* (17) *is locally determinate in the regions* S_{a}^{a} *and* S_{b}^{a}*.*

(a) *When* $\omega > 2$

$$S_{\mathrm{a}}^{a} = \Big\{(\delta,\, \kappa) \in \mathbb{R}_{+}^{2} \,|\, \omega/(\omega-1) < \delta \leq \hat{\delta},\, \kappa > 0 \cup \delta > \hat{\delta},\, 0 < \kappa \leq \kappa^{*}(\delta)\Big\}$$

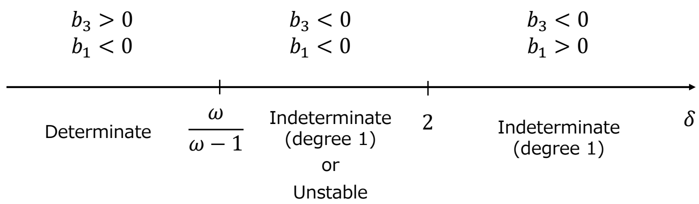
(b) *When* $\omega < 2$

$$S_{\mathrm{b}}^{a} = \big\{(\delta,\, \kappa) \in \mathbb{R}_{+}^{2} \,|\, \delta > \omega/(\omega-1),\, 0 < \kappa \leq \kappa^{*}(\delta)\big\}$$

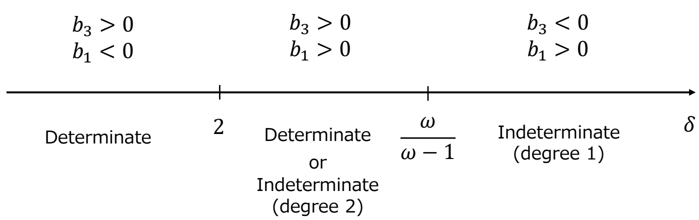
Proposition 3.1(a) indicates that an active monetary policy ($\kappa > 0$) must be accompanied by a passive fiscal policy ($\delta > 1$), a necessary condition. This conclusion is actually consistent with that of previous studies, including Leeper (1991), and not particularly new. Notably, when $0 < \kappa \leq 1$, a sufficiently large δ (i.e., fiscal policy is sufficiently passive) is a sufficient condition for achieving local determinacy, but when $\kappa > 1$, such a policy stance leads to indeterminacy (degree 2). Thus, when monetary policy is active, κ has a certain threshold, based on which, the policy stance on fiscal policy δ must be determined. If $\kappa \leq 1$, local determinacy will always be achieved if the fiscal policy is sufficiently passive; however, if $\kappa > 1$, for the achievement of local determinacy, it must be "appropriately" passive.

Proposition 3.1(b) indicates that, if κ is less than 1, local determinacy can be achieved by increasing δ (having a sufficiently passive fiscal policy). This is no different from the results obtained when $\omega > 2$. However, if κ exceeds 1, regardless of the value of δ (regardless of the fiscal policy), it will not be possible to achieve determinacy. This differs from the case in which $\omega > 2$.

Following Tsuzuki (2016), if we set $\psi = 1$ and set β and $\bar{\tau}$ at 0.2 and 0.12, respectively, which can be considered typically appropriate values, we get $\omega \gtreqless 2$ for $\sigma \gtreqless 0.833$. Here, $\omega > 2$ means that the elasticity of consumption ($1/\sigma$) is less than $1/0.833 \approx 1.2$, and $\omega < 2$ means it is greater than 1.2 because in (2), σ stands for the reciprocal of the elasticity of consumption with respect to currency holdings. Therefore, these results signify that in an economy where the elasticity of consumption with respect to currency holdings is less than 1.2, a "moderately" passive fiscal policy ($\omega/(\omega-1) < \delta \leq (\kappa^{*})^{-1}(\kappa)$) is required when monetary policy is sufficiently active ($\kappa \geq 1$) (Proposition 3.1(a)). In an economy in which the elasticity of consumption with respect to currency holdings exceeds 1.2, no appropriate fiscal policy controls exist (Proposition 3.1(b)).



(a) When $\omega > 2$



(b) When $\omega < 2$

Figure 4 The signs of b_3 and b_1 under passive monetary policy

3.4 Passive monetary policy

In this section, we assume that $\kappa < 0$ and consider cases in which monetary policy is passive. As in active cases, the signs of b_3 and b_1 depend only on δ. The conditions for the sign of b_1 are given by (25). The conditions for the sign of b_3 can be rewritten as

$$b_3 \lesseqgtr 0 \iff \delta \gtreqless \frac{\omega}{\omega - 1}. \tag{29}$$

From the correlation between (25) and (29), we obtain Figure 4 under Assumption 3.2. The nature of the system can be roughly divided into two cases: $\omega > 2$ and $\omega < 2$.

In the case of $\omega > 2$, from Figures 2(a) and 4(a), we can clarify the nature of dynamics, as shown in Figure 3(a). The area of local determinacy is set at S_{a}^{p}, represented by the shaded area in Figure 3(a). When $\omega < 2$, Figure 3(b) is obtained from Figures 2(b) and 4(b). The area of local determinacy when $\omega < 2$ corresponds to the part denoted by shaded area S_{b}^{p} in Figure 3(b).

From this discussion, we obtain the following proposition.

Proposition 3.2 *Under Assumptions* 3.1 *and* 3.2 *and a passive monetary policy, the equilibrium of System* (17) *is locally determinate in the areas* S_{a}^{p} *and* S_{b}^{p}.

(a) *When* $\omega > 2$ *and* (b) *when* $\omega < 2$

$$S_{\mathrm{a}}^{p} = S_{\mathrm{b}}^{p} = \left\{(\delta,\, \kappa) \in \mathbb{R}^2 \,|\, 0 < \delta < \omega/(\omega - 1),\, \kappa < 0\right\}$$

As indicated in Proposition 3.2, under a passive monetary policy, the condition on δ that leads to local determinacy ($0 < \delta < \omega/(\omega - 1)$) in case (a) fully conforms with that in case (b). Therefore, compared with the former case ($\omega > 2$), the latter case ($\omega < 2$) has a numerically broader interval of δ in which determinacy is achieved.

The models by Leeper (1991) and Tsuzuki (2016), which assumed lump-sum taxes instead of income taxes, correspond to our model in the case in which the component in row 2, column 3 of the Jacobian matrix J is zero (i.e., $\tau_a = 0$).[23] In addition, the component in row 3 and column 3 of J is $\rho - \tau_\ell$, where τ_ℓ represents the extent to which the lump-sum tax reacts to changes in the government debt. In this case, the matrix J becomes decomposable, and the characteristic equation is

$$\Delta(\lambda) = \left[\lambda^2 - \rho\lambda + P_1\,(R_v - 1)\,y^*\right]\left[\lambda - (\rho - \tau_\ell)\right] = 0.$$

The three roots of this equation are derived with the following:

$$\lambda_1,\, \lambda_2 = \frac{\rho}{2}\left[1 \pm \sqrt{1 - \frac{4P_1\,(R_v - 1)\,y^*}{\rho^2}}\right],\ \lambda_3 = \rho - \tau_\ell.$$

Thus, if $R_v > 1$ then $\mathrm{Re}\,(\lambda_1) > 0$ and $\mathrm{Re}\,(\lambda_2) > 0$, and if $R_v < 1$ then $\lambda_1 > 0$ and $\lambda_2 < 0$. Also, if $\tau_\ell > \rho$ then $\lambda_3 < 0$, and if $\tau_\ell < \rho$ then $\lambda_3 > 0$. Active fiscal policy is defined as $\tau_\ell < \rho$ and passive fiscal policy as $\tau_\ell > \rho$. Therefore, in the case of lump-sum taxes, the conclusion is extremely simple: when monetary policy is active ($R_v > 1$), fiscal policy must be passive ($\tau_\ell > \rho$), and when monetary policy is passive ($R_v < 1$), fiscal policy must be active ($\tau_\ell < \rho$). These are the necessary and sufficient conditions for local determinacy. However, in our model ($\tau_a > 0$ and $\tau_\ell = 0$), having a passive fiscal policy with an active monetary policy is merely a necessary condition for local determinacy, as indicated in Proposition 3.1. If $\delta > \omega/(\omega - 1) \Leftrightarrow \tau_a > \frac{\rho}{y^*}\frac{\omega}{\omega-1}$ is not satisfied, we cannot obtain local determinacy. Simultaneously, if monetary policy is passive, an active fiscal policy is a sufficient condition for local determinacy, as indicated in Proposition 3.2. Although fiscal policy is passive, local determinacy can be achieved if $\delta < \omega/(\omega - 1) \Leftrightarrow \tau_a < \frac{\rho}{y^*}\frac{\omega}{\omega-1}$. Accordingly, we cannot infer that if monetary policy is passive, fiscal policy must be active.

4 Inflation and output targeting

We now discuss situations in which both the rate of inflation and the production volume are considered as variables in the monetary authorities' policy targets.

[23] In the case of lump-sum taxes, the second equation in (14), derived from the NK Phillips function (9), becomes independent of $\tau(t)$, so the component in row 2, column 3 of the Jacobian matrix J is zero.

4.1 Characteristic equation

If inflation and output targeting are implemented simultaneously, the monetary policy rule can be rewritten as

$$R(t) = \bar{R} + R_y \left[y(t) - y^*\right] + R_v \left[v(t) - v^*\right], \tag{30}$$

where $R_y \geq 0$ is a policy parameter that expresses the degree to which the nominal interest rate responds to changes in production.

Substituting Equations (16) and (30) into System (14) yields the following differential equations.

$$\begin{aligned}
\dot{y}(t) &= \left[\bar{R} + R_y \left\{y(t) - y^*\right\} + R_v \left\{v(t) - v^*\right\} - v(t) - \rho\right] y(t), \\
\dot{v}(t) &=\rho \left[v(t) - v^*\right] - \frac{\phi}{\eta} y(t)^{1+\psi} + \frac{\sigma\left(\phi - 1\right)}{\eta\left(1 - \beta\right)} \left[1 - \bar{\tau} - \tau_a \left\{a(t) - a^*\right\}\right], \\
\dot{a}(t) &= \left[\bar{R} + R_y \left\{y(t) - y^*\right\} + R_v \left\{v(t) - v^*\right\} - v(t)\right] a(t) - \frac{1-\beta}{\sigma} y(t) \\
&\quad - \left[\bar{\tau} + \tau_a \left\{a(t) - a^*\right\}\right] y(t) + \beta y(t).
\end{aligned} \tag{31}$$

The steady-state values of System (31) are given in (18). The Jacobian matrix of System (31) evaluated at the steady-state point is

$$\tilde{J} = \begin{bmatrix} R_y y^* & \left(R_v - 1\right) y^* & 0 \\ -P_1 & \rho & -P_2 \tau_a \\ \left(R_y - \rho/y^*\right) a^* & \left(R_v - 1\right) a^* & \rho - \tau_a y^* \end{bmatrix}.$$

We assume that $\tilde{J}$ is a nonsingular matrix, that is, $\det \tilde{J} \neq 0$.

The characteristic equation is

$$\tilde{\Delta}(\lambda) := \left|\lambda I - \tilde{J}\right| = \lambda^3 + \tilde{b}_1 \lambda^2 + \tilde{b}_2 \lambda + \tilde{b}_3 = 0,$$

where

$$\tilde{b}_1 = -\mathrm{tr}\, \tilde{J} = \rho\left(\delta - 2 - \gamma\right), \tag{32}$$

$$\tilde{b}_2 = \left|\tilde{J}_{11}\right| + \left|\tilde{J}_{22}\right| + \left|\tilde{J}_{33}\right| = \rho^2 \left[1 - \delta + \kappa\left(\omega + \delta\right) - \left(\delta - 2\right)\gamma\right], \tag{33}$$

$$\tilde{b}_3 = -\det \tilde{J} = \rho^3 \left[\kappa \left\{\delta\left(\omega - 1\right) - \omega\right\} + \left(\delta - 1\right)\gamma\right], \tag{34}$$

and

$$\gamma := \frac{y^*}{\rho} R_y.$$

Furthermore,

$$\begin{aligned}
\tilde{b}_1 \tilde{b}_2 - \tilde{b}_3 =&\rho^3 \left[\left\{\delta^2 - \left(1 + \gamma\right)\delta - \left(1 + \gamma\right)\omega\right\}\kappa\right. \\
&\left. - \left(1 + \gamma\right)\left\{\delta^2 - \left(3 + \gamma\right)\delta + 2\left(1 + \gamma\right)\right\}\right].
\end{aligned} \tag{35}$$

The model in the previous section excludes the case in which $R_v = 1$ by using the assumption that $\det J \neq 0$. However, in this model, there is no contradiction between $R_v = 1$ and $\det \tilde{J} \neq 0$. When $R_v = 1$, $\tilde{J}$ becomes decomposable, and the characteristic equation is

$$\tilde{\Delta}(\lambda) = (R_y y^* - \lambda)(\rho - \lambda)(\rho - \tau_a y^* - \lambda) = 0.$$

Therefore, we have three characteristic roots, $\lambda_1 = R_y y^* > 0$, $\lambda_2 = \rho > 0$, and $\lambda_3 = \rho - \tau_a y^*$. If fiscal policy is active ($\tau_a < \rho/y^*$), then $\lambda_3 > 0$; conversely, if fiscal policy is passive ($\tau_a > \rho/y^*$), then $\lambda_3 < 0$.[24] Therefore, the equilibrium is unstable under an active fiscal policy and is locally determinate under a passive fiscal policy. Next, we analyze cases in which $R_v \neq 1$ ($\kappa \neq 0$).

4.2 Consideration of the signs of the coefficients

The analytical method is basically the same as in the previous section. We simply clarify the sets (δ, κ) that give the sign boundaries for $\tilde{b}_3$, for $\tilde{b}_1$, and for $\tilde{b}_1\tilde{b}_2 - \tilde{b}_3$.

4.2.1 The sign of $\tilde{b}_3$

To depict $\tilde{b}_3 = 0$ on plane δ-κ, solving this equation for κ gives the following.

$$\kappa = \tilde{\kappa}^{**}(\delta) := \frac{(1-\delta)\gamma}{\delta(\omega - 1) - \omega}. \tag{36}$$

For all $\delta > 0$, $\tilde{\kappa}^{**}(\delta)$ is an increasing function. Moreover, $\tilde{\kappa}^{**}(0) = -\gamma/\omega$ and $\tilde{\kappa}^{**}(1) = 0$.

The two asymptotes of the fractional function (36) are given by

$$\delta = \frac{\omega}{\omega - 1}, \tag{37}$$

$$\kappa = -\frac{\gamma}{\omega - 1}. \tag{38}$$

Therefore, the function $\tilde{\kappa}^{**}(\delta)$ can be depicted as in Figure 5. The differences between Figures 5(a) and 5(b) are discussed later.

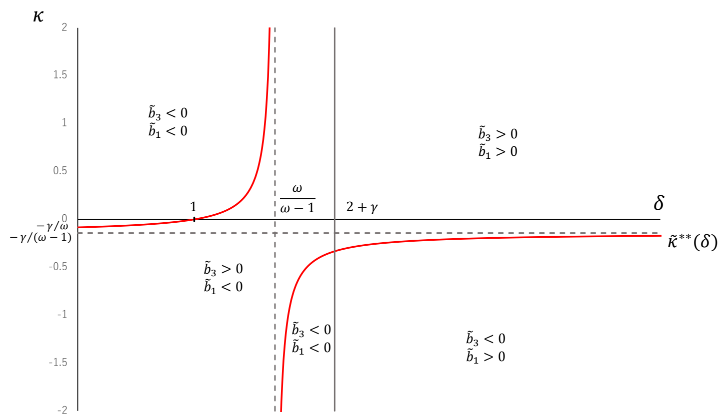
In the case of $\delta < \omega/(\omega - 1)$ (left side of the asymptote), from Equation (34), we know that $\tilde{b}_3$ is decreasing with respect to κ. Thus, $\tilde{b}_3 < 0$ holds for $\kappa > \tilde{\kappa}^{**}(\delta)$, and $\tilde{b}_3 > 0$ holds for $\kappa < \tilde{\kappa}^{**}(\delta)$. As for $\delta > \omega/(\omega - 1)$ (on the right side of the asymptote), $\tilde{b}_3 > 0$ for $\kappa > \tilde{\kappa}^{**}(\delta)$ and $\tilde{b}_3 < 0$ for $\kappa < \tilde{\kappa}^{**}(\delta)$. Next, we consider the sign of $\tilde{b}_1$.

4.2.2 The sign of $\tilde{b}_1$

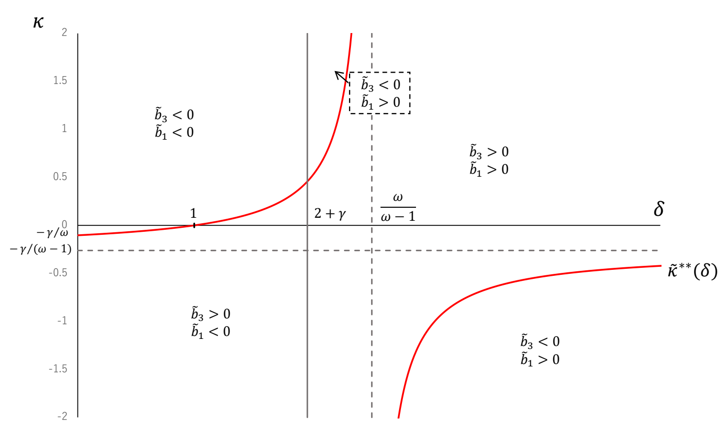
From (32), we get the following correlations.

$$\tilde{b}_1 \lesseqgtr 0 \iff \delta \lesseqgtr 2 + \gamma.$$

[24] The assumption $\det \tilde{J} \neq 0$ is not valid when $\tau_a = \rho/y^*$.



(a) Case (I): $\omega > \frac{2+\gamma}{1+\gamma}$



(b) Case (II): $\omega < \frac{2+\gamma}{1+\gamma}$

Figure 5 The graph of $\tilde{\kappa}^{**}(\delta)$ and the signs of $\tilde{b}_3$ and $\tilde{b}_1$

As the sign of $\tilde{b}_1$ does not depend on κ, the graph of $\tilde{b}_1 = 0$ depicts a vertical line on the plane δ-κ. The region to the left of the vertical line is the area of $\tilde{b}_1 < 0$, and to the right is that $\tilde{b}_1 > 0$. The structure of the area of determinacy will differ significantly depending on whether the line $\delta = 2 + \gamma$ is to the left or to the right of the line (37), which is an asymptote of (36). The scenario is divided into two cases.

(I) $\frac{\omega}{\omega-1} < 2 + \gamma$ or equivalently, $\omega > \frac{2+\gamma}{1+\gamma}$.

(II) $\frac{\omega}{\omega-1} \geq 2 + \gamma$ or equivalently, $\omega \leq \frac{2+\gamma}{1+\gamma}$.

Figure 5(a) depicts case (I) and Figure 5(b) depicts case (II) (note that this depicts the case in which (II) holds with strict inequality). In these figures, the sign combinations for $\tilde{b}_3$ and $\tilde{b}_1$ are

(A) $\tilde{b}_3 > 0$ and $\tilde{b}_1 \leq 0$,

(B) $\tilde{b}_3 < 0$ and $\tilde{b}_1 \geq 0$,

(C) $\tilde{b}_3 > 0$ and $\tilde{b}_1 > 0$,

(D) $\tilde{b}_3 < 0$ and $\tilde{b}_1 < 0$.

In the area where (A) holds true, the Routh–Hurwitz criterion is not satisfied; thus, the equilibrium is locally determinate. Where (B) holds true, the inverse Routh–Hurwitz criterion is not satisfied; thus, the equilibrium is locally indeterminate (degree 1). As the signs of the characteristic roots are not specified in areas where (C) or (D) hold true, we need to consider the sign of $\tilde{b}_1\tilde{b}_2 - \tilde{b}_3$.

4.2.3 The sign of $\tilde{b}_1\tilde{b}_2-\tilde{b}_3$

From (35), solving $\tilde{b}_1\tilde{b}_2 - \tilde{b}_3 = 0$ for κ gives us

$$\kappa = \tilde{\kappa}^*(\delta) := \frac{(1+\gamma)\left[\delta^2 - (3+\gamma)\,\delta + 2\,(1+\gamma)\right]}{\delta^2 - (1+\gamma)\,\delta - (1+\gamma)\,\omega}. \tag{39}$$

We depict this function on the plane δ-κ.

The sign of $\tilde{\kappa}^{*\prime}(\delta)$ is the exact same sign as the following quadratic function if the denominator of (39) is non-zero.

$$\tilde{f}(\delta) := \delta^2 - (1+\gamma)\,(\omega+2)\,\delta + (1+\gamma)\left(1+\gamma+\frac{3+\gamma}{2}\omega\right).$$

The solutions to equation $\tilde{f}(\delta) = 0$ can be derived with the following.

$$\tilde{\delta}^*_{\pm} = \frac{1}{2}\,(1+\gamma)\left[(\omega+2) \pm \omega\sqrt{1 - \frac{2\,(1-\gamma)}{(1+\gamma)\,\omega}}\right].$$

From the above calculations, we can derive the properties of the function $\tilde{\kappa}^*(\delta)$.

(i) When $\omega > \frac{2(1-\gamma)}{1+\gamma}$, $\tilde{\kappa}^*(\delta)$ is an increasing function for $[0,\, \tilde{\delta}^*_-)$ and $(\tilde{\delta}^*_+,\, \infty)$. For $(\tilde{\delta}^*_-,\, \tilde{\delta}^*_+)$, it is a decreasing function.

(ii) When $\omega < \frac{2(1-\gamma)}{1+\gamma}$, $\tilde{f}(\delta) > 0$ holds for all $\delta > 0$. Therefore, $\tilde{\kappa}^*(\delta)$ is an increasing function for all $\delta > 0$.

Under Assumption 3.2, ω must be greater than 1, so if $\gamma \geq 1/3$, case (ii) cannot occur. In addition, the case in which $\omega = \frac{2(1-\gamma)}{1+\gamma}$ can be analyzed as a specific case of $\omega > \frac{2(1-\gamma)}{1+\gamma}$, so we omit it from our analysis here and move it to Appendix A.4.

The asymptote parallel to the vertical axis (κ axis) of the function $\tilde{\kappa}^*(\delta)$ can be derived with the following equation:

$$\delta = \tilde{\tilde{\delta}} := \frac{1}{2}\left(1+\gamma\right)\left(1+\sqrt{1+\frac{4\omega}{1+\gamma}}\right). \tag{40}$$

In case (i), in which $\omega > \frac{2(1-\gamma)}{1+\gamma}$ holds, we have $\tilde{\delta}^*_- < \tilde{\tilde{\delta}} < \tilde{\delta}^*_+$ because $\tilde{f}(\tilde{\tilde{\delta}}) < 0$.[25] Furthermore, as $\tilde{\kappa}^*(\delta) = C$ (C is an arbitrary constant) is a quadratic function, the number of δ satisfying this equation is at most two. This means that $\tilde{\kappa}^*(\tilde{\delta}^*_+) \geq \tilde{\kappa}^*(\tilde{\delta}^*_-)$ holds true.[26]

Moreover, $\tilde{\kappa}^*(0) = -2\left(1+\gamma\right)/\omega$, $\lim_{\delta\to\infty} \tilde{\kappa}^*(\delta) = 1+\gamma$, and $\tilde{\kappa}^*(1+\gamma) = \tilde{\kappa}^*(2) = 0$ also hold true.[27] Additionally, in case (i), $1+\gamma < \tilde{\delta}^*_- < 2 < \tilde{\tilde{\delta}}$ is valid, and in case (ii), $1+\gamma < \tilde{\tilde{\delta}} < 2$ is valid.

From the above discussion, in cases of (i) $\omega > \frac{2(1-\gamma)}{1+\gamma}$ and (ii) $\omega < \frac{2(1-\gamma)}{1+\gamma}$, the graph of function (39) is depicted as in Figures 6(a) and 6(b), respectively. Note that we depicted the case where $1+\gamma < 2$ in case (i). Even if $1+\gamma > 2$, however, the shape of the curves remains the same.

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[25]We show that $\tilde{f}(\tilde{\tilde{\delta}}) \lesseqqgtr 0$ when $\omega \gtreqqless \frac{2(1-\gamma)}{1+\gamma}$. Using the equation $\tilde{\tilde{\delta}}^2 = (1+\gamma)\left(\tilde{\tilde{\delta}}+\omega\right)$, we can rewrite the expression $\tilde{f}(\tilde{\tilde{\delta}})$ as

$$\begin{aligned}\tilde{f}(\tilde{\tilde{\delta}}) &= (1+\gamma)\left(\tilde{\tilde{\delta}}+\omega\right) - (1+\gamma)(\omega+2)\,\tilde{\tilde{\delta}} + (1+\gamma)\left(1+\gamma+\frac{3+\gamma}{2}\omega\right) \\ &= (1+\gamma)\left[-(\omega+1)\,\tilde{\tilde{\delta}} + \left(1+\gamma+\frac{5+\gamma}{2}\omega\right)\right].\end{aligned}$$

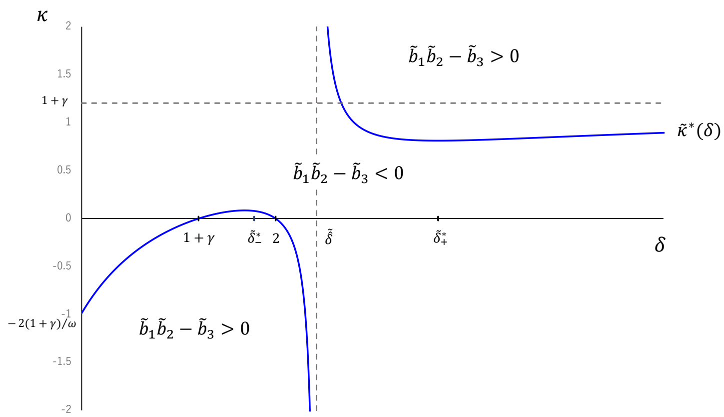
Substitution of (40) in the above expression yields the following.

$$\tilde{f}(\tilde{\tilde{\delta}}) = \frac{1}{2}(1+\gamma)^2\sqrt{1+\frac{4\omega}{1+\gamma}}\left[\sqrt{1+\frac{4\omega}{1+\gamma}} - (1+\omega)\right].$$

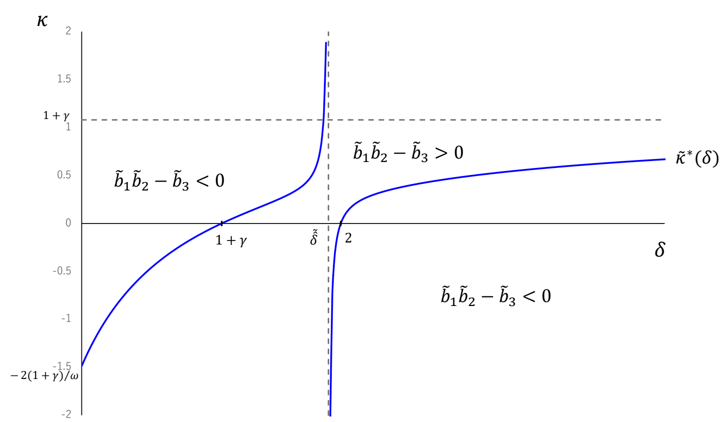
Therefore, $\tilde{f}(\tilde{\tilde{\delta}}) \lesseqqgtr 0 \iff \omega \gtreqqless \frac{2(1-\gamma)}{1+\gamma}$.

[26]If $\tilde{\kappa}^*(\tilde{\delta}^*_+) < \tilde{\kappa}^*(\tilde{\delta}^*_-)$, then four values of δ satisfying $\tilde{\kappa}^*(\delta) = C$ exist for $C \in \left(\tilde{\kappa}^*(\tilde{\delta}^*_+),\, \tilde{\kappa}^*(\tilde{\delta}^*_-)\right)$.

[27]Factorizing the numerator of $\tilde{\kappa}^*(\delta)$, it becomes $(1+\gamma)\left[\delta-(1+\gamma)\right](\delta-2)$. Therefore, $1+\gamma$ and 2 are the roots of the equation $\tilde{\kappa}^*(\delta) = 0$.



(a) Case (i): $\omega > \frac{2(1-\gamma)}{1+\gamma}$



(b) Case (ii): $\omega < \frac{2(1-\gamma)}{1+\gamma}$

Figure 6 The graph of $\tilde{\kappa}^*(\delta)$ and the signs of $\tilde{b}_1\tilde{b}_2 - \tilde{b}_3$

Next, we clarify the sign of $\tilde{b}_1\tilde{b}_2-\tilde{b}_3$ in each region of the plane δ-κ. In Figure 6, when $\delta < \tilde{\hat{\delta}}$ (left side of the asymptote), $\delta^2-(1+\gamma)\,\delta-(1+\gamma)\,\omega<0$ holds. In this case, from (35), $\tilde{b}_1\tilde{b}_2-\tilde{b}_3$ is decreasing with respect to κ. Therefore, $\tilde{b}_1\tilde{b}_2-\tilde{b}_3<0$ for $\kappa>\tilde{\kappa}^*(\delta)$ and $\tilde{b}_1\tilde{b}_2-\tilde{b}_3>0$ for $\kappa<\tilde{\kappa}^*(\delta)$; the reverse holds when $\delta<\tilde{\hat{\delta}}$ (right side of the asymptote). Thus, we obtained the following.

- If $\delta<\tilde{\hat{\delta}}$ and $\kappa<\tilde{\kappa}^*(\delta)$, or $\delta>\tilde{\hat{\delta}}$ and $\kappa>\tilde{\kappa}^*(\delta)$, then $\tilde{b}_1\tilde{b}_2-\tilde{b}_3>0$.
- If $\delta<\tilde{\hat{\delta}}$ and $\kappa>\tilde{\kappa}^*(\delta)$, or $\delta>\tilde{\hat{\delta}}$ and $\kappa<\tilde{\kappa}^*(\delta)$, then $\tilde{b}_1\tilde{b}_2-\tilde{b}_3<0$.

We combine Figures 5 and 6. Since $2+\gamma>2\,(1-\gamma)$, in case (I) in which $\omega>\frac{2+\gamma}{1+\gamma}$ is valid (Figure 5(a)), case (i) where $\omega>\frac{2(1-\gamma)}{1+\gamma}$ (Figure 6(a)) must always apply. It can also be shown that there are the relationships $\frac{\omega}{\omega-1}\lesseqgtr 2+\gamma \iff \tilde{\hat{\delta}} \gtreqless 2+\gamma$. Therefore, in case (I), the following relationships are established.

$$1<\frac{\omega}{\omega-1}<2+\gamma<\tilde{\hat{\delta}}<\tilde{\delta}^*_+,\ \text{and}\ 1+\gamma<\tilde{\delta}^*_-<2\ (\text{if}\ 1+\gamma<2).$$

The magnitude relationship between $1+\gamma$ (or 2) and $\omega/(\omega-1)$ is not established. However, this is not essential for discussion. Furthermore, the magnitude relationship between $\tilde{\delta}^*_-$ and $\omega/(\omega-1)$ is also not verified. However, this point does not affect our discussion either. Meanwhile, in case (II), represented by Figure 5(b), where $\omega\le\frac{2+\gamma}{1+\gamma}$ holds, both cases of (i) $\omega>\frac{2(1-\gamma)}{1+\gamma}$ and (ii) $\omega<\frac{2(1-\gamma)}{1+\gamma}$ in Figure 6 are possible. In other words, there are following two specific cases: (II-i) $\frac{2(1-\gamma)}{1+\gamma}<\omega\le\frac{2+\gamma}{1+\gamma}$ and (II-ii) $\omega<\frac{2(1-\gamma)}{1+\gamma}$. In case (II-i), we obtain the following relationships.

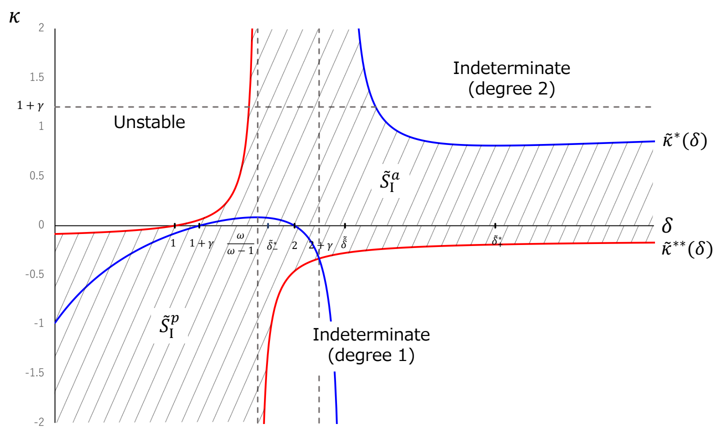
$$1<1+\gamma<\tilde{\delta}^*_-<2<\tilde{\hat{\delta}}<2+\gamma<\frac{\omega}{\omega-1},\ \text{and}\ \tilde{\hat{\delta}}<\tilde{\delta}^*_+\ (\text{if}\ 1+\gamma<2).$$

In case (II-ii), we have

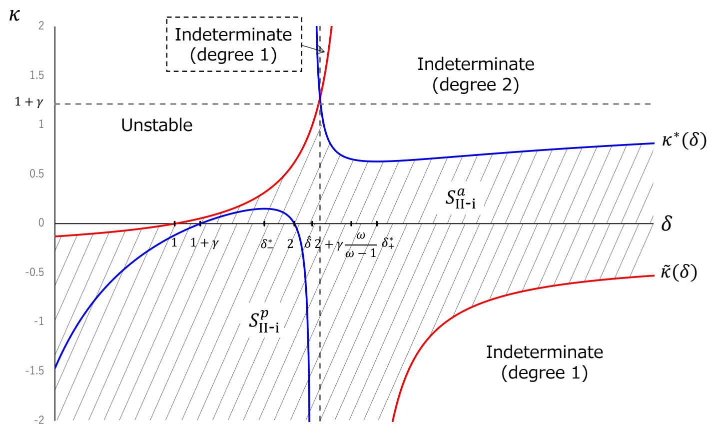
$$1<1+\gamma<\tilde{\hat{\delta}}<2<2+\gamma<\frac{\omega}{\omega-1}.$$

Furthermore, we can verify that $\tilde{\kappa}^*(2+\gamma)=\tilde{\kappa}^{**}(2+\gamma)$ holds, regardless of the case type. Considering these results, Figures 5 and 6 can be combined to obtain Figure 7. Figure 7(a) shows case (I), and Figures 7(b) and 7(c) show cases (II-i) and (II-ii), respectively.

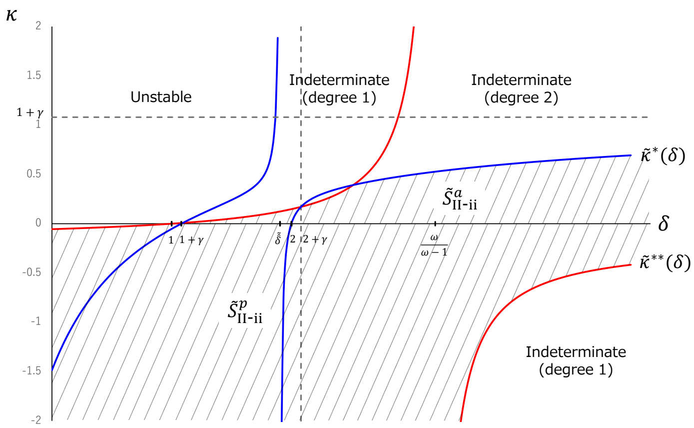
We can now elicit the local determinacy of the two cases of (C) $\tilde{b}_3>0$ and $\tilde{b}_1>0$ and of (D) $\tilde{b}_3<0$ and $\tilde{b}_1<0$ in Subsection 4.2.2. In (C), if $\tilde{b}_1\tilde{b}_2-\tilde{b}_3>0$, then all the conditions of the Routh–Hurwitz criterion are satisfied; thus, the equilibrium is indeterminate (degree 2). Conversely, if $\tilde{b}_1\tilde{b}_2-\tilde{b}_3\le 0$, then one of the conditons of this criterion is not satisfied, so equilibrium is determinate. At the same time, in (D), as $\tilde{b}_1\tilde{b}_2-\tilde{b}_3<0$ holds true, all inverse Routh–Hurwitz conditions are satisfied, and thus the equilibrium is unstable. Thus, in the case



(a) Case (I): $\omega > \frac{2+\gamma}{1+\gamma}$



(b) Case (II-i): $\frac{2(1-\gamma)}{1+\gamma} < \omega \leq \frac{2+\gamma}{1+\gamma}$



(c) Case (II-ii): $\omega < \frac{2(1-\gamma)}{1+\gamma}$

Figure 7 Areas of local determinacy when inflation and output targeting are implemented

of active monetary policy, the areas of local determinacy are $\tilde{S}^a_{\text{I}}$, $\tilde{S}^a_{\text{II-i}}$, and $\tilde{S}^a_{\text{II-ii}}$, as indicated by the diagonal lines in Figures 7(a) to 7(c).[28]

Let us summarize our results in a proposition.

Proposition 4.1 *Under Assumptions* 3.1 *and* 3.2 *and an active monetary policy, the equilibrium of System* (31) *is locally determinate in the areas* $\tilde{S}^a_{\text{I}}$, $\tilde{S}^a_{\text{II-i}}$, *and* $\tilde{S}^a_{\text{II-ii}}$.

Case (I) $\omega > \frac{2+\gamma}{1+\gamma}$

$$\tilde{S}^a_{\text{I}} = \left\{ (\delta,\,\kappa) \in \mathbb{R}^2_+ \;\middle|\; \begin{array}{l} 1 < \delta < \omega/\left(\omega - 1\right),\, 0 < \kappa < \tilde{\kappa}^{**}(\delta) \\ \cup\, \omega/\left(\omega - 1\right) \le \delta \le \hat{\hat{\delta}},\, \kappa > 0 \\ \cup\, \delta > \hat{\hat{\delta}},\, 0 < \kappa \le \tilde{\kappa}^{*}(\delta) \end{array} \right\}$$

Case (II-i) $\frac{2(1-\gamma)}{1+\gamma} < \omega \le \frac{2+\gamma}{1+\gamma}$ *and Case* (II-ii) $\omega < \frac{2(1-\gamma)}{1+\gamma}$

$$\tilde{S}^a_{\text{II-i}} = \tilde{S}^a_{\text{II-ii}} = \left\{ (\delta,\,\kappa) \in \mathbb{R}^2_+ \;\middle|\; \begin{array}{l} 1 < \delta \le 2 + \gamma,\, 0 < \kappa < \tilde{\kappa}^{**}(\delta) \\ \cup\, \delta > 2 + \gamma,\, 0 < \kappa < \min\left[\tilde{\kappa}^{*}(\delta),\, \tilde{\kappa}^{**}(\delta)\right] \end{array} \right\}$$

If the monetary policy is passive, the shaded areas $\tilde{S}^p_{\text{I}}$, $\tilde{S}^p_{\text{II-i}}$, and $\tilde{S}^p_{\text{II-ii}}$ in Figure 7 represent the locally determinate areas. The results are summarized as follows.

[28] In Cases (I) and (II-i) as well as Case (II-ii), when $\omega < (1+\gamma)/(4\gamma)$, the curves $\tilde{\kappa}^*(\delta)$ and $\tilde{\kappa}^{**}(\delta)$ have two intersections at points other than $\delta = 2 + \gamma$. In such cases, the region enclosed by the two curves may exhibit indeterminacy of degree 1 or instability. However, the basic structure of the determinate region does not differ greatly from that in Figure 7, so we will omit those cases here.

Proposition 4.2 *Under Assumptions* 3.1 *and* 3.2 *and a passive monetary policy, the equilibrium of System* (31) *is locally determinate in the areas* $\tilde{S}^p_{\text{I}}$, $\tilde{S}^p_{\text{II-i}}$, *and* $\tilde{S}^p_{\text{II-ii}}$.

Case (I) $\omega > \frac{2+\gamma}{1+\gamma}$, *Case* (II-i) $\frac{2(1-\gamma)}{1+\gamma} < \omega \leq \frac{2+\gamma}{1+\gamma}$, *and Case* (II-ii) $\omega < \frac{2(1-\gamma)}{1+\gamma}$

$$\tilde{S}^p_{\text{I}} = \tilde{S}^p_{\text{II-i}} = \tilde{S}^p_{\text{II-ii}} = \left\{ (\delta,\, \kappa) \in \mathbb{R}^2 \;\middle|\; \begin{array}{l} 0 < \delta \leq 1,\, \kappa < \tilde{\kappa}^{**}(\delta) \\ \cup\, 1 < \delta \leq \omega/\left(\omega - 1\right),\, \kappa < 0 \\ \cup\, \delta > \omega/\left(\omega - 1\right),\, \tilde{\kappa}^{**}(\delta) < \kappa < 0 \end{array} \right\}$$

Let us compare Figure 3(a) (pure inflation targeting) and Figure 7(a) (inflation and output targeting). When $\gamma = 0$ ($R_y = 0$), these figures are completely in agreement, so we can make a direct comparison. The areas of determinacy are larger in Figure 7(a) than in 3(a). Therefore, output targeting expands the determinacy area. This result is consistent with Bullard and Mitra (2002) and others.[29]

Comparing Figure 7(a) with Figure 7(b) or Figure 7(a) with Figure 7(c), similar to the case of pure inflation targeting in the previous section, in case (I), there definitely exists an interval of δ that achieves determinacy for an arbitrary value of κ. However, in cases (II-i) and (II-ii), there is no such interval for $\kappa > 1 + \gamma$. Even if $\omega < 2$, it could apply to Figure 7(a). This is because if γ is sufficiently large, the inequality $\omega > \frac{2+\gamma}{1+\gamma}$ in case (I) can hold true even if $\omega < 2$. The situation where $\omega < 2$ assuming pure inflation targeting is represented in Figure 3(b). Therefore, under the conditions depicted in Figure 3(b), if the monetary authorities are sufficiently aggressive in output targeting, the circumstances change to those depicted in Figure 7(a) and we can eliminate the case in which the fiscal policy is ineffective. We consider this a new finding that demonstrates the effectiveness of output targeting.

From Proposition 4.2, we see that under a passive monetary policy, no differences exist in the structures of the determinate areas in all three cases. This outcome contrasts with that when the monetary policy is active (Proposition 4.1).

Furthermore, in Figures 7(a), 7(b), and 7(c), an unstable area exists not only in areas where monetary policy is active, but also where it is passive. In the previous section's model, which assumed pure inflation targeting, an unstable area appeared only for active monetary policy (see Figures 3(a) and 3(b)). This difference has a policy implication. The unstable area when monetary policy is passive exists in an active fiscal policy area ($\delta < 1$). Therefore, monetary authorities seeking to prevent instability must determine whether to implement output targeting depending on whether the fiscal policy is active or passive when monetary policy (inflation targeting) is passive. If the fiscal policy is active (passive), not implementing (implementing) output targeting increases the probability of avoiding instability.

[29]See Section 3 in Bullard and Mitra (2002) and Chapter 4, Subsection 4.3 in Galí (2015).

5 Conclusion

This study employs a three-dimensional NK model that incorporates income tax, a distortionary tax, to theoretically analyze how the interaction between monetary policy (inflation and output targeting) and fiscal policy (government debt targeting) affects the local determinacy of the equilibrium path.[30]

First, we have considered cases in which the monetary authorities implement pure inflation targeting (when $\gamma = 0$). In these cases, the effects of monetary and fiscal policies greatly differ depending on the value of a constant ω, which consists of the structural parameters. If $\omega > 2$, then under an active monetary policy ($\kappa > 0$), a fiscal policy ($\delta > \omega/(\omega - 1)$) will always achieve local determinacy (Figure 3(a)). However, if $\omega < 2$, there could be situations in which determinacy cannot be achieved under an active monetary policy, regardless of the type of fiscal policy pursued (Figure 3(b)). In other words, κ has a threshold value ($\kappa = 1$). If $\kappa \leq 1$, local determinacy is achieved when $\delta > \omega/(\omega - 1)$. However, if $\kappa > 1$, there can be no value of δ that achieves the same result. At the same time, under passive monetary policy ($\kappa < 0$), regardless of the value of ω, there will always be an appropriate control for δ (Figures 3(a) and 3(b)).

In the case of pure inflation targeting, having a passive fiscal policy ($\delta > 1$) under an active monetary policy is merely a necessary condition for local determinacy. Similarly, an active fiscal policy ($\delta < 1$) under a passive monetary policy is a sufficient condition, not a necessary condition, for local determinacy. Therefore, according to our model, if monetary policy is active, fiscal policy must be passive. However, we cannot state that fiscal policy must be active if monetary policy is passive.

Next, we have considered cases in which the monetary authorities manipulate the nominal interest rate in response to changes in the inflation rate and output following the so-called inflation and output targeting policy. In this case, if monetary policy is active, when at least $\omega > 2$ (condition $\omega > \frac{2+\gamma}{1+\gamma}$ of case (I) holds true), the increase in the response to output will broaden the area of determinacy (comparing Figure 3(a) with Figure 7(a)). This finding is consistent with that of prior research, and not new.

However, if $\omega < 2$, even if output targeting is implemented, and it is not so proactive (to the extent that condition $\omega \leq \frac{2+\gamma}{1+\gamma}$ of case (II) holds true), it could give rise to a situation similar to that of pure inflation targeting, where there is no appropriate δ (Figures 7(b) and 7(c)). If output targeting is proactive enough (to the extent that condition $\omega > \frac{2+\gamma}{1+\gamma}$ of case (I) holds true), even if $\omega < 2$, such a situation will not occur (Figure 7(a)). Thus, we can eliminate cases in which the fiscal policy is rendered ineffective if monetary authorities implement output targeting under an active monetary policy. This is a new finding that demonstrates the effectiveness of output targeting, not covered in prior research. By contrast, under a passive monetary policy ($\kappa < 0$), regardless of the value of

[30] All discussions here are related to local equilibrium determinacy in the vicinity of a steady-state point in System (14). Appendix A.5 describes how we apply the bifurcation theorem to make several points regarding global determinacy.

ω, an appropriate control for δ always exists (Figures 7(a) to 7(c)). This finding is the same as that for pure inflation targeting.

Under a passive monetary policy, an unstable area does not appear when pure inflation targeting is assumed (Figure 3), but appears only when an inflation- and output-targeting policy is assumed (Figure 7). This unstable area occurs where fiscal policy is active ($\delta < 1$). Therefore, if monetary authorities seeking to avoid instability, practice a passive monetary policy (passive inflation targeting), the decision to implement output targeting will depend on whether the fiscal policy is active or passive. If the fiscal policy is active (or passive), the decision to not implement (or implement) output targeting will increase the likelihood of avoiding instability.

A Appendices

A.1 Dynamic optimization of household–firm units

From the dynamic optimizing behavior of a representative household–firm unit, we derive Equations (8) to (10).[31]

Pontryagin's maximum principle

Using Equations (2) and (4) to (7), we can establish the following current-value Hamiltonian for the problem to be solved by the household–firm unit.

$$\begin{aligned}
&\mathcal{H}\left(c_j,\, m_j,\, p_j,\, p,\, y,\, \tau,\, R,\, v_j,\, a_j,\, \mu_1,\, \mu_2\right)\\
&\quad := \log\left(c_j(t)^{\sigma} m_j(t)\right) - \frac{1}{1+\psi}\left[\left(\frac{p_j(t)}{p(t)}\right)^{-\phi} y(t)\right]^{1+\psi} - \frac{\eta}{2}\left(v_j(t) - v_j^*\right)^2\\
&\quad + \mu_1(t)\left[(1-\tau(t))\left(\frac{p_j(t)}{p(t)}\right)^{1-\phi} y(t) + r(t)a_j(t) - c_j(t) - R(t)m_j(t)\right]\\
&\quad + \mu_2(t)v_j(t)p_j(t),
\end{aligned}$$

where $\mu_1(t)$ and $\mu_2(t)$ are co-state variables of the state variables $a_j(t)$ and $p_j(t)$, respectively.

Pontryagin's maximum principle (necessary conditions for maximizing the path value U expressed in (1)) comprises motion Equations (5) and (7), as well as the following three conditions:

[31] See Chapter 8 in Chiang (1992) for details on the methods of dynamic optimization used here.

1. $\max_{c_j,\, m_j,\, v_j} \mathcal{H}$:

$$\frac{\partial \mathcal{H}}{\partial c_j} = \sigma \frac{1}{c_j(t)} - \mu_1(t) = 0, \tag{A.1}$$

$$\frac{\partial \mathcal{H}}{\partial m_j} = \frac{1}{m_j(t)} - \mu_1(t) R(t) = 0, \tag{A.2}$$

$$\frac{\partial \mathcal{H}}{\partial v_j} = -\eta \left(v_j(t) - v_j^*\right) + \mu_2(t) p_j(t) = 0. \tag{A.3}$$

2. Motion equations of the co-state variables are

$$\dot{\mu}_1(t) = \rho\mu_1(t) - \frac{\partial \mathcal{H}}{\partial a_j} = \rho\mu_1(t) - r(t)\mu_1(t), \tag{A.4}$$

$$\begin{aligned}\dot{\mu}_2(t) = \rho\mu_2(t) - \frac{\partial \mathcal{H}}{\partial p_j} =& \rho\mu_2(t) - \phi \frac{y_j(t)^{1+\psi}}{p_j(t)} \\ &- \mu_1(t)\left(1-\tau(t)\right)(1-\phi)\frac{y_j(t)}{p(t)} - \mu_2(t) v_j(t).\end{aligned} \tag{A.5}$$

3. Transversality conditions are

$$\text{(i)} \lim_{t\to\infty} e^{-\rho t}\mu_1(t) = 0, \ \text{(ii)} \lim_{t\to\infty} e^{-\rho t}\mu_2(t) = 0, \ \text{(iii)} \lim_{t\to\infty} e^{-\rho t}\mathcal{H}(t) = 0. \tag{A.6}$$

Under Condition 1, the Hamiltonian $\mathcal{H}$ must be maximized with respect to control variables c_j, m_j, and v_j. As $\mathcal{H}$ is non-linear with respect to c_j, m_j and v_j, (A.1)–(A.3) are the first-order conditions for establishing Condition 1. The Hessian determinant of $\mathcal{H}(c_j,\, m_j,\, v_j)$ is expressed as

$$|H| := \begin{vmatrix} -\sigma\frac{1}{c_j^2} & 0 & 0 \\ 0 & -\frac{1}{m_j^2} & 0 \\ 0 & 0 & -\eta \end{vmatrix}.$$

This gives

$$|H_1| = -\sigma\frac{1}{c_j^2} < 0, \ |H_2| = \sigma\frac{1}{(c_j m_j)^2} > 0, \ |H_3| = -\eta\,|H_2| < 0,$$

where $|H_k|$ denotes the kth leading principal minor of $|H|$. Therefore, the second-order conditions are also satisfied. Condition 2 describes the motion of the co-state variables $\mu_1(t)$ and $\mu_2(t)$. The equations in Condition 3 are known as transversality conditions and are designed to remove divergent solutions. These equations show that if $\mu_1(t)$, $\mu_2(t)$, and $\mathcal{H}(t)$ are all finite values when $t \to \infty$, Condition 3 will be satisfied. If the steady-state point $(c^*,\, v^*,\, a^*)$ is not unstable, Equations (i)–(iii) are established. Confirm this: (i) according to (A.1), if the set of solution paths $c_j(t)$ is bounded, the set of $\mu_1(t)$ is also bounded; and (ii) according to (A.3), if the set of the solution paths $v_j(t)$ is bounded, the set of $\mu_2(t)p_j(t)$ is also bounded. Therefore, if $\lim_{t\to\infty} p_j(t) = \infty$, then $\lim_{t\to\infty} \mu_2(t) = 0$, and if $\lim_{t\to\infty} p_j(t) \neq \infty$, then $\lim_{t\to\infty} \mu_2(t) =$ finite. (iii) At the steady-state point, both $r(t) = \bar{R} - v^*$ and $\tau(t) = \bar{\tau}$ are constants. As shown in (i), if $c_j(t)$ is bounded, then $\mu_1(t)$ is bounded, and in this case, according to (A.2), $m_j(t)$ is also bounded. Furthermore, according to (12) and (13), as $(1-\beta)\,y(t) = c(t)$, if $c(t)$ is bounded, $y(t)$ is also bounded. Finally, as $p(t) = p_j(t)$ in a symmetric equilibrium, $p_j(t)/p(t)$ equals 1. Therefore, if the set of solution paths for $a_j(t)$, which is the only remaining variable comprising $\mathcal{H}$, is bounded, then $\mathcal{H}$ is also bounded.

From (A.1), we obtain $\dot{\mu}_1(t)/\mu_1(t) = -\dot{c}_j(t)/c_j(t)$. If we substitute this into Equation (A.4), we obtain the consumption Euler Equation (8). From (A.3), we obtain $\mu_2(t) = \eta\left(v_j(t) - v_j^*\right)/p_j(t)$ and $\dot{\mu}_2(t) = (\eta/p_j(t))\left[\dot{v}_j(t) - v_j(t)\left(v_j(t) - v_j^*\right)\right]$. If we substitute these into (A.5) and use (A.1) to eliminate $\mu_1(t)$, we obtain the NK Phillips function in (9). Finally, we obtain the money demand function (10) from (A.1) and (A.2).

Testing Arrow's condition as a sufficient condition

We test Arrow's condition as a sufficient condition for maximizing the path value U.[32] Substituting the optimal control (candidate) $(c_j(t),\, m_j(t),\, v_j(t))$ obtained from (A.1) to (A.3) in the Hamiltonian gives us

$$\begin{aligned}\mathcal{H}^0(a_j,\, p_j) :=& \sigma \log\left(\frac{\sigma}{\mu_1(t)}\right) + \log\left(\frac{1}{\mu_1(t)R(t)}\right) - \frac{1}{1+\psi}\left[p(t)^{\phi} y(t) p_j(t)^{-\phi}\right]^{1+\psi} \\ &+ \mu_1(t)\left[(1-\tau(t))\, p(t)^{\phi-1} p_j(t)^{1-\phi} y(t) + r(t)a_j(t) - \frac{1+\sigma}{\mu_1(t)}\right] \\ &+ v_j^* \mu_2(t) p_j(t) + \frac{\mu_2(t)^2}{2\eta} p_j(t)^2.\end{aligned}$$

Given the values of $\mu_1(t)$ and $\mu_2(t)$, if this "maximized Hamiltonian" $\mathcal{H}^0$ is jointly concave with respect to $a_j(t)$ and $p_j(t)$ for all $t \geq 0$, then the maximum principle is sufficient.[33]

Let $\left|\mathcal{H}^{0a}\right|$ be the Hessian determinant of $\mathcal{H}^0$ in the order $(a_j,\, p_j)$. Then the principal minors of $\left|\mathcal{H}^{0a}\right|$ are calculated as $\left|\mathcal{H}_1^{0a}\right| = \left|\mathcal{H}_2^{0a}\right| = 0$. If the order is

[32] See Chapters 8 and 9 in Chiang (1992).

[33] Assuming that $\tilde{a}_j(t)$ and $\tilde{p}_j(t)$ are optimal state paths and that $a_j(t)$ and $p_j(t)$ are other feasible paths, the sufficient conditions include the conditions of $\lim_{t\to\infty} \mu_1(t)e^{-\rho t}\left[a_j(t) - \tilde{a}_j(t)\right] \geq 0$ and $\lim_{t\to\infty} \mu_2(t)e^{-\rho t}\left[p_j(t) - \tilde{p}_j(t)\right] \geq 0$. However, if we restrict the feasible paths to be in the neighborhood of the optimal path, they will be satisfied with strict equality by the transversality conditions (A.6).

$(p_j,\, a_j)$, the principal minors are

$$\begin{aligned}\left|\mathcal{H}_1^{0p}\right| =& -\phi\left[\phi\left(1+\psi\right)+1\right]\left(p(t)^{\phi}y(t)\right)^{1+\psi}p_j(t)^{-\phi(1+\psi)-2}\\ &+\phi\mu_1(t)\left(1-\tau(t)\right)\left(\phi-1\right)p(t)^{\alpha-1}y(t)p_j(t)^{-\phi-1}+\frac{\mu_2(t)^2}{\eta},\\ \left|\mathcal{H}_2^{0p}\right| =& 0.\end{aligned}$$

In view of the symmetry among household–firm units and using (12) and (13), we can rewrite $\left|\mathcal{H}_1^{0p}\right|$ as

$$\left|\mathcal{H}_1^{0p}\right| = \left\{-\phi\left[\phi\left(1+\psi\right)+1\right]y(t)^{1+\psi}+\frac{\phi\left(\phi-1\right)\left(1-\tau(t)\right)\sigma}{1-\beta}+\eta\left(v(t)-v^*\right)^2\right\}p(t)^{-2}.$$

If this value is non-positive for all t, then $\mathcal{H}^0$ is concave and Arrow's sufficient condition is satisfied. Although the sign of this expression is not usually set, we can verify that the condition is satisfied near the steady-state point expressed in (18), regardless of the parameter values.

A.2 Inverse Routh–Hurwitz criterion

We present the necessary and sufficient conditions (inverse Routh–Hurwitz criterion) for the real parts of all the roots to be positive in a linear continuous-time system.[34]

The components of the matrix and the signs of the roots

We consider the following linear continuous-time system.

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}, \tag{A.7}$$

where A is an $n \times n$ matrix and $\boldsymbol{x}$ is a vector of order n. We assume $|A| \neq 0$. Although all the components of the vector are functions of time t, we omit these notations for simplification. The characteristic equation of this system is expressed as

$$|\lambda_A I - A| = 0,$$

where λ_A denotes the eigenvalue of A, and I is a unit matrix.

Also let

$$B = -A,$$

then $|B| = (-1)^n |A| \neq 0$. The characteristic equation of the system represented by matrix B is

$$|\lambda_B I - B| = 0,$$

[34] This proof is based on Asada, Flaschel, and Proaño (2007). Incidentally, the necessary and sufficient conditions for the absolute value of all roots in a linear discrete, rather than continuous, time system to be less than 1 is known as the Schur–Cohn criterion. Asada (2013) proved the inverse Schur–Cohn criterion (the necessary and sufficient conditions for the absolute value of all roots to be greater than 1).

where λ_B is the eigenvalue of B.

We get the following for any n.

$$\begin{aligned} |\lambda_B I - B| &= |\lambda_B I - (-A)| \\ &= (-1)^n \left| -\lambda_B I - A \right| = 0. \end{aligned}$$

Therefore, $\lambda_A = -\lambda_B$ is true. In other words, if the real parts of all roots of A are negative, then the real parts of all roots of $B = -A$ are positive (necessary and sufficient).

Coefficient criteria

Matrix A represents the following.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Using this expression, the characteristic equation of System (A.7) is written as

$$|\lambda_A I - A| = \lambda_A^n + a_1 \lambda_A^{n-1} + a_2 \lambda_A^{n-2} + \cdots + a_{n-1} \lambda_A + a_n = 0,$$

where

$$
\begin{aligned}
a_1 &= -\operatorname{tr} A = -a_{11} - a_{22} - \cdots - a_{nn}, \\
a_2 &= \text{Sum of all second order principal minors of } |A| \\
&= \begin{vmatrix} a_{n-1,\,n-1} & a_{n-1,\,n} \\ a_{n,\,n-1} & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{n-2,\,n-2} & a_{n-2,\,n} \\ a_{n,\,n-2} & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \\
&\vdots \\
a_{n-1} &= (-1)^{n-1} \times (\text{Sum of all } n-1\text{-th order principal minors of } |A|) \\
&= (-1)^{n-1} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2,\,n-1} & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3,\,n-1} & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,\,2} & a_{n-1,\,3} & \cdots & a_{n-1,\,n-1} & a_{n-1,\,n} \\ a_{n2} & a_{n3} & \cdots & a_{n,\,n-1} & a_{nn} \end{vmatrix} \\
&+ (-1)^{n-1} \begin{vmatrix} a_{11} & a_{13} & \cdots & a_{1,\,n-1} & a_{1n} \\ a_{31} & a_{33} & \cdots & a_{3,\,n-1} & a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1,\,1} & a_{n-1,\,3} & \cdots & a_{n-1,\,n-1} & a_{n-1,\,n} \\ a_{n1} & a_{n3} & \cdots & a_{n,\,n-1} & a_{nn} \end{vmatrix} \\
&+ \cdots + (-1)^{n-1} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,\,n-2} & a_{1,\,n-1} \\ a_{21} & a_{22} & \cdots & a_{2,\,n-2} & a_{2,\,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-2,\,1} & a_{n-2,\,2} & \cdots & a_{n-2,\,n-2} & a_{n-2,\,n-1} \\ a_{n-1,\,1} & a_{n-1,\,2} & \cdots & a_{n-1,\,n-2} & a_{n-1,\,n-1} \end{vmatrix}, \\
a_n &= (-1)^n \times \det A = (-1)^n \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.
\end{aligned}
$$

At the same time, matrix B is expressed as

$$
B = -A = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & -a_{nn} \end{bmatrix}.
$$

The characteristic equation of a system represented by B is

$$
|\lambda_B I - B| = \lambda_B^n + b_1 \lambda_B^{n-1} + b_2 \lambda_B^{n-2} + \cdots + b_{n-1}\lambda_B + b_n = 0,
$$

where

$$
\begin{aligned}
b_1 &= -\operatorname{tr} B = a_{11} + a_{22} + \cdots + a_{nn} = -a_1,\\
b_2 &= \begin{vmatrix} -a_{n-1,\,n-1} & -a_{n-1,\,n} \\ -a_{n,\,n-1} & -a_{nn} \end{vmatrix} + \begin{vmatrix} -a_{n-2,\,n-2} & -a_{n-2,\,n} \\ -a_{n,\,n-2} & -a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix}\\
&= \begin{vmatrix} a_{n-1,\,n-1} & a_{n-1,\,n} \\ a_{n,\,n-1} & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{n-2,\,n-2} & a_{n-2,\,n} \\ a_{n,\,n-2} & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_2,\\
&\ \ \vdots\\
b_{n-1} &= (-1)^{n-1} \begin{vmatrix} -a_{22} & -a_{23} & \cdots & -a_{2,\,n-1} & -a_{2n} \\ -a_{32} & -a_{33} & \cdots & -a_{3,\,n-1} & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-1,\,2} & -a_{n-1,\,3} & \cdots & -a_{n-1,\,n-1} & -a_{n-1,\,n} \\ -a_{n2} & -a_{n3} & \cdots & -a_{n,\,n-1} & -a_{nn} \end{vmatrix}\\
&\quad + (-1)^{n-1} \begin{vmatrix} -a_{11} & -a_{13} & \cdots & -a_{1,\,n-1} & -a_{1n} \\ -a_{31} & -a_{33} & \cdots & -a_{3,\,n-1} & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-1,\,1} & -a_{n-1,\,3} & \cdots & -a_{n-1,\,n-1} & -a_{n-1,\,n} \\ -a_{n1} & -a_{n3} & \cdots & -a_{n,\,n-1} & -a_{nn} \end{vmatrix}\\
&\quad + \cdots + (-1)^{n-1} \begin{vmatrix} -a_{11} & -a_{12} & \cdots & -a_{1,\,n-2} & -a_{1,\,n-1} \\ -a_{21} & -a_{22} & \cdots & -a_{2,\,n-2} & -a_{2,\,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -a_{n-2,\,1} & -a_{n-2,\,2} & \cdots & -a_{n-2,\,n-2} & -a_{n-2,\,n-1} \\ -a_{n-1,\,1} & -a_{n-1,\,2} & \cdots & -a_{n-1,\,n-2} & -a_{n-1,\,n-1} \end{vmatrix}\\
&\quad = (-1)^{n-1} a_{n-1},\\
b_n &= (-1)^n \times \det B = (-1)^{2n} \times \det A = (-1)^n a_n.
\end{aligned}
$$

Therefore, the following correlations hold true:[35]

$$
\begin{aligned}
b_1 &= -a_1,\\
b_2 &= a_2,\\
&\ \vdots\\
b_{n-1} &= (-1)^{n-1} a_{n-1},\\
b_n &= (-1)^n a_n.
\end{aligned}
$$

When $n = 3$

For $n = 3$, the necessary and sufficient conditions for the real parts of all roots of A to be negative (the Routh–Hurwitz criterion) are $a_1 > 0$, $a_3 > 0$, and

[35] When the subscripts are odd numbers, the signs are reversed; when they are even, they remain as is.

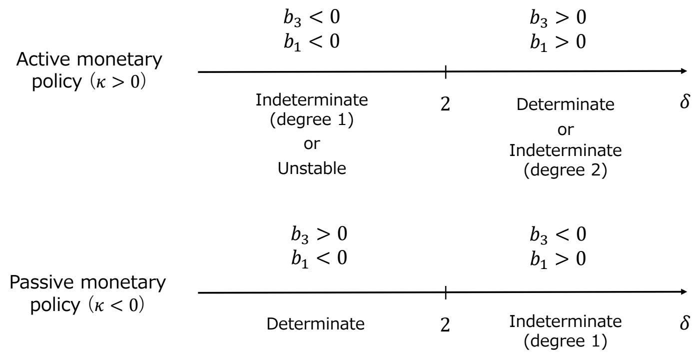


Figure A.1 The signs of b_3 and b_1 when $\omega = 2$

$a_1 a_2 - a_3 > 0$.[36]

$\Longrightarrow$ The necessary and sufficient conditions for the real parts of all roots of B to be positive (the inverse Routh–Hurwitz criterion) to be met are $b_1 < 0$, $b_3 < 0$, and $b_1 b_2 - b_3 < 0$ ($\because$ $a_1 a_2 - a_3 = (-b_1) b_2 - (-b_3) = -b_1 b_2 + b_3$).[37]

A.3 Analysis of identical threshold scenarios

Regarding the signs of b_3 and b_1 when $\omega = 2$, if monetary policy is active, these are shown in the upper part of Figure A.1 from (24) and (25). If monetary policy is passive, they are shown in the lower part of Figure A.1 from (29) and (25). In the chart in the lower part of Figure A.1, the local determinacy of the equilibrium under passive monetary policy is determined regardless of the sign of $b_1 b_2 - b_3$. Thus, if $\delta < 2$, equilibrium is locally determinate, and if $\delta > 2$, it is indeterminate (degree 1). However, in the case of an active monetary policy, we need to test the sign of $b_1 b_2 - b_3$.

When $\omega = 2$, if $\delta = 2$, then $b_1 b_2 - b_3 \equiv 0$ holds for any value of κ. Therefore, in the case of $\omega = 2$, in plane δ-κ, the line standing perpendicular at $\delta = 2$ stands for $b_1 b_2 - b_3 = 0$. However, if $\delta = 2$, then as $b_3 = 0$, according to the assumption of $\det J \neq 0$, we must exclude points on this line from consideration. In addition, if $\omega = 2$, then $\hat{\delta} = 2$.

Furthermore, if $\omega = 2$, function (26) that represents $b_1 b_2 - b_3 = 0$ can be rewritten as

$$\kappa^*(\delta) = \frac{(\delta - 1)(\delta - 2)}{(\delta + 1)(\delta - 2)}. \tag{A.8}$$

[36]See, for instance, Chapter 16 in Gandolfo (2010). The original Routh–Hurwitz criterion also required that $a_2 > 0$. We have omitted it here because $a_2 > 0$ is always satisfied when the three conditions given here are satisfied.

[37]Although $b_2 > 0$ must hold true, this condition is included in the three conditions given here.

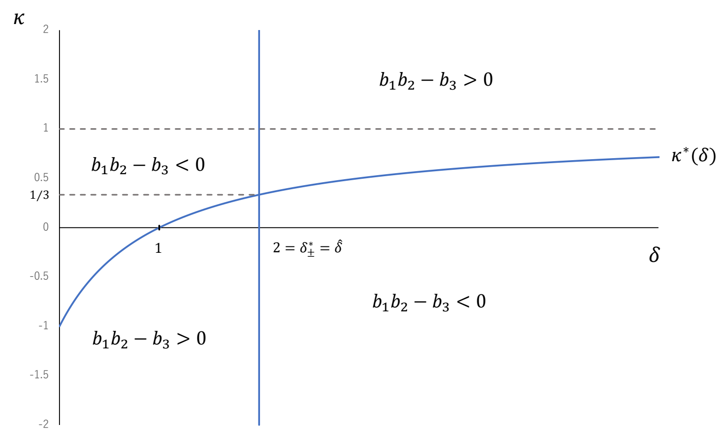


Figure A.2 The graph of $\kappa^*(\delta)$ and the signs of $b_1b_2 - b_3$ when $\omega = 2$

Thus, (A.8) can be divided into the following two cases.[38]

$$\kappa = \kappa^*(\delta) = \begin{cases} \dfrac{\delta-1}{\delta+1} & \text{if } \delta \neq 2. \\ \dfrac{1}{3} & \text{if } \delta = 2. \end{cases} \tag{A.9}$$

The graph of function (A.8) on plane δ-κ is shown in Figure A.2. $\kappa = \frac{\delta-1}{\delta+1}$ is the upward-sloping curve asymptotic to 1, and $\kappa^*(0) = -1$ and $\kappa^*(1) = 0$ are satisfied. The equations $\delta^*_- = 2$ and $\delta^*_+ = 2$ hold true, which indicate that the two extremal points $\left(\delta^*_-,\, \kappa(\delta^*_-)\right)$ and $\left(\delta^*_+,\, \kappa(\delta^*_+)\right)$ of the curve $\kappa^*(\delta)$ are aggregated at a single point $\left(\hat{\delta},\, 1/3\right)$ (see Figure 2(a)).

The method for evaluating the sign of $b_1b_2 - b_3$ in each area of Figure A.2 is exactly the same as that for when $\omega \neq 2$ discussed in the main text.

From Figures A.1 and A.2, we determine the areas of local determinacy in the plane δ-κ to be those delineated by the shaded areas in Figure A.3. In the case of active monetary policy, if $\kappa < 1$, local determinacy can be achieved through the appropriate choice of δ. However, if $\kappa \geq 1$, such a choice does not exist. At the same time, in the case of passive monetary policy, it is locally determinate as long as $\delta < 2$.

A.4 Analysis of critical cases

As indicated in Subsection 4.2.3, the equation $\omega = \frac{2(1-\gamma)}{1+\gamma}$ is possible only in case (II) where $\omega \leq \frac{2+\gamma}{1+\gamma}$ with the condition that $\gamma < \frac{1}{3}$. The signs of $\tilde{b}_3$ and $\tilde{b}_1$ are shown in Figure 5(b). Below, we present the graph representing $\tilde{b}_1\tilde{b}_2 - \tilde{b}_3 = 0$.

[38] $\kappa = 1/3$ is obtained by applying l'Hôpital's rule to (A.8).

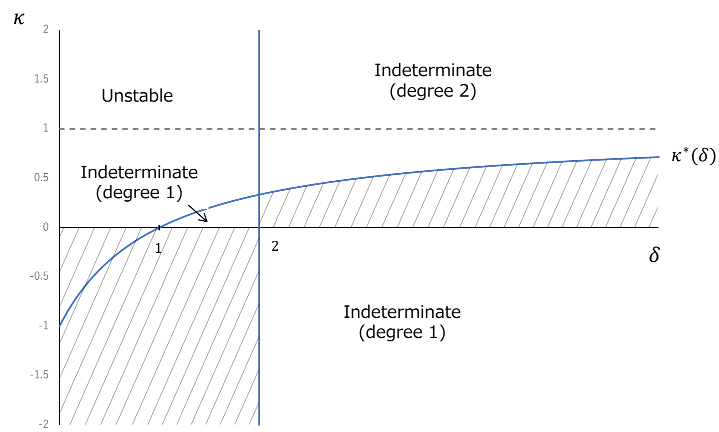


Figure A.3 Areas of local determinacy when $\omega = 2$

When $\omega = \frac{2(1-\gamma)}{1+\gamma}$, if $\delta = 2$, for any value of κ, $\tilde{b}_1\tilde{b}_2 - \tilde{b}_3 \equiv 0$ will hold. Therefore, on the plane δ-κ, $\tilde{b}_1\tilde{b}_2 - \tilde{b}_3 = 0$ represents the line standing perpendicular at $\delta = 2$. When $\omega = \frac{2(1-\gamma)}{1+\gamma}$, $\tilde{\tilde{\delta}} = 2$ is also true.

Furthermore, if $\omega = \frac{2(1-\gamma)}{1+\gamma}$, function (39) representing $\tilde{b}_1\tilde{b}_2 - \tilde{b}_3 = 0$ can be rewritten as

$$\tilde{\kappa}^*(\delta) = \frac{(1+\gamma)\left[\delta - (1+\gamma)\right](\delta - 2)}{\left[\delta + (1-\gamma)\right](\delta - 2)}. \tag{A.10}$$

Therefore, we get[39]

$$\kappa = \tilde{\kappa}^*(\delta) = \begin{cases} \dfrac{(1+\gamma)\left[\delta - (1+\gamma)\right]}{\delta + (1-\gamma)} & \text{if } \delta \neq 2. \\ \dfrac{-\gamma^2+1}{3-\gamma} & \text{if } \delta = 2. \end{cases}$$

Figure A.4 graphically depicts (A.10) on the plane δ-κ. $\kappa = \frac{(1+\gamma)[\delta-(1+\gamma)]}{\delta+(1-\gamma)}$ is the upward sloping curve that is asymptotic to $1+\gamma$, and $\tilde{\kappa}^*(0) = \frac{(1+\gamma)^2}{\gamma-1}$ and $\tilde{\kappa}^*(1+\gamma) = 0$ are satisfied. The equations $\tilde{\delta}^*_- = 2$ and $\tilde{\delta}^*_+ = 2$ hold true, which indicate that the two points $\left(\tilde{\delta}^*_-, \tilde{\kappa}^*(\tilde{\delta}^*_-)\right)$ and $\left(\tilde{\delta}^*_+, \tilde{\kappa}^*(\tilde{\delta}^*_+)\right)$ are aggregated into one point $\left(2, \frac{-\gamma^2+1}{3-\gamma}\right)$ (see Figure 6(b)).

The method for evaluating the sign of $\tilde{b}_1\tilde{b}_2 - \tilde{b}_3$ in each area is exactly the same as that for $\omega \neq \frac{2(1-\gamma)}{1+\gamma}$ in the main text.

Considering the relationship of $1 < 1+\gamma < 2 < 2+\gamma < \omega/(\omega-1)$, from Figures 5(b) and A.4, we determine the areas of local determinacy in the plane

[39] $\kappa = \frac{-\gamma^2+1}{3-\gamma}$ is obtained from applying the l'Hôpital rule to (A.10).

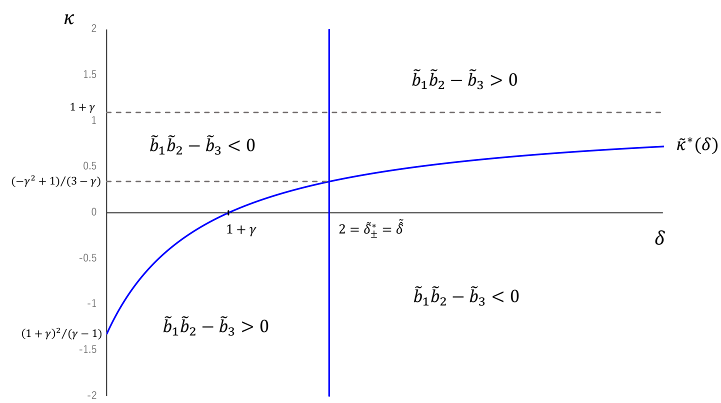


Figure A.4 The graph of $\tilde{\kappa}^*(\delta)$ and the signs of $\tilde{b}_1\tilde{b}_2 - \tilde{b}_3$ when $\omega = \frac{2(1-\gamma)}{1+\gamma}$

δ-κ to be those delineated by the shaded area in Figure A.5. With an active monetary policy, if $\kappa < 1 + \gamma$, local determinacy can be achieved through the appropriate choice of δ. However, if $\kappa \geq 1 + \gamma$, such a choice does not exist. At the same time, with a passive monetary policy, a value of δ that achieves determinacy is present for all cases of $\kappa < 0$.

A.5 Additional remarks concerning global determinacy

The entire discussion so far has concerned the local determinacy of equilibrium in the vicinity of the steady-state point in System (14). This section is still within the realm of local theory, but presents discussions on global determinacy using a bifurcation theorem.

If a closed loop exists around a steady-state point and is a stable cycle (limit cycle), the equilibrium is locally determinate, but globally indeterminate. This is because the equilibrium path is indeterministic as an orbit asymptotic to the periodic solution is not uniquely established.[40] Similarly, although the steady-state point is locally unstable, the equilibrium is globally indeterminate if surrounded by a limit cycle.

Benhabib, Schmitt-Grohé, and Uribe (2003) assumed a non-linear interest rate rule to demonstrate that a limit cycle occurs around one among multiple steady-state points.[41] In their proof, they employed Hopf bifurcation theorem. In our model, a closed loop can exist around the steady-state point because of the non-linearity of the NK Phillips function in System (14) (although this steady-state point is unique). Below, we take δ as the bifurcation parameter

[40] As the orbit asymptotic to the periodic solution satisfies the transversality conditions, it does not conflict with dynamic optimization of household–firm units.

[41] Benhabib, Schmitt-Grohé and Uribe (2003) analytically demonstrated that a periodic solution exists and used the Kopell–Howard theorem to indicate the presence of a homoclinic orbit, as well as indicate that a saddle connection could occur.

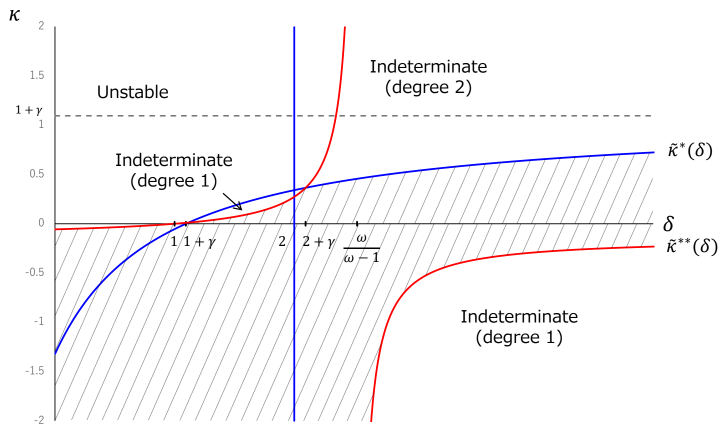


Figure A.5 Locally determinate areas when $\omega = \frac{2(1-\gamma)}{1+\gamma}$

and explore the possibility that a periodic solution exists for the case of pure inflation targeting.

The Hopf bifurcation theorem, expressed in terms of the coefficients of the characteristic equation, includes the following conditions:[42]

Lemma A.1 *Suppose that System* (17) *satisfies the following conditions at* $\delta = \delta_h$.

(I) $b_1(\delta_h) \neq 0$, $b_2(\delta_h) > 0$, $b_1(\delta_h)b_2(\delta_h) - b_3(\delta_h) = 0$.

(II) $\left.\frac{d(b_1(\delta)b_2(\delta)-b_3(\delta))}{d\delta}\right|_{\delta=\delta_h} \neq 0$.

Then, there exists a periodic solution with period approximately $2\pi/\operatorname{Im}\lambda(\delta_h)$ *that bifurcates from steady-state point* (18).

We investigate these conditions using Figure 3(a). The condition concerning $b_1(\delta)$ in (I) is satisfied if $\delta \neq 2$. With regard to $b_2(\delta)$, although our discussion has not directly focused on the sign of this, if we use (21) to find the function that represents $b_2(\delta) = 0$, we obtain the following.

$$\kappa = \kappa_2(\delta) := \frac{\delta - 1}{\omega + \delta},$$

where

$$\kappa_2(0) = -\frac{1}{\omega} > \kappa^*(0) = -\frac{2}{\omega},\ \kappa_2'(\delta) = \frac{\omega + 1}{(\omega + \delta)^2} > 0,$$
$$\lim_{\delta \to \infty} \kappa_2(\delta) = 1, \text{ and } \kappa_2(1) = 0.$$

[42]See Asada and Semmler (1995).

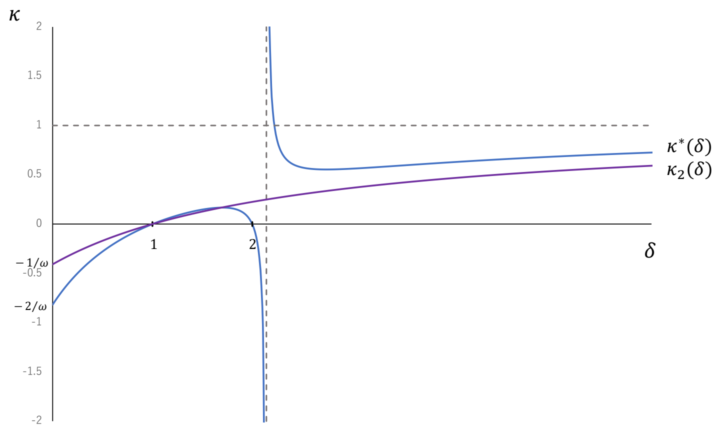


Figure A.6 The graph of $\kappa_2(\delta)$

Therefore, in plane δ-κ, function $\kappa_2(\delta)$ is depicted as in Figure A.6. $b_2(\delta) > 0$ is true above the curve. The equation $b_1 b_2 - b_3 = 0$, which is the third condition in (I), holds true everywhere on the curve $\kappa^*(\delta)$ depicted in Figure 3(a) or Figure A.6. Lastly, condition (II) is met when δ crosses the curve $\kappa^*(\delta)$.

Accordingly, only in the case of an active monetary policy does the bifurcation value δ_h that meets all conditions in Lemma A.1 exist on curve $\kappa^*(\delta)$. However, the Hopf bifurcation theorem does not describe the stabilities of periodic solutions. For instance, suppose that in Figure 3(a), δ decreases and passes δ_h (i.e., it crosses curve $\kappa^*(\delta)$), and the equilibrium changes from locally indeterminate (degree 2) to a state of determinacy. If a periodic solution exists for $\delta < \delta_h$ (the steady-state point is locally determinate), then the Hopf bifurcation is termed supercritical bifurcation and the periodic solution is stable (a limit cycle). By contrast, if a periodic solution exists for $\delta > \delta_h$ (the steady-state point is locally indeterminate), it is called subcritical bifurcation, and the periodic solution is unstable. Therefore, in the former case, there is global indeterminacy. Determining which case applies requires converting System (17) into a center manifold.[43] However, as we would need to specify the parameters to perform such a calculation, we do not go that far.[44]

[43]Subsection 3.2 in Guckenheimer and Holmes (1983) presents details regarding conversion to a center manifold.

[44]For example, Subsection 3.2.2 in Lorenz (1993) examines the conversion to a center manifold and the stability of the periodic solution in a two-dimensional system. The study shows that periodic solution stability depends on the third-order partial derivative of the non-linear term in the canonical form.

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